Chapter VII
Trees and Large Cardinals in L

In this chapter we concentrate on the notion of a κ-tree in the case where κ is an inaccessible cardinal. In this case, assuming $V = L$, both the notion of a κ-Souslin tree and of a κ-Kurepa tree turn out to be closely related to large cardinal properties. Thus this chapter extends both Chapter IV, where we studied κ^+-trees, and (parts of) Chapter V, where we dealt with large cardinals.

1. Weakly Compact Cardinals and κ-Souslin Trees

The notion of a weakly compact cardinal has already been introduced in V.1, and we refer the reader back there for basic definitions. In particular, V.1.3 gives several equivalent definitions of weak compactness, and V.1.5 proves the result, relevant to us here, that if κ is a weakly compact cardinal, then $[\kappa$ is weakly compact$]^L$.

Assuming $V = L$, we shall prove that if κ is an inaccessible cardinal, then κ is weakly compact iff there is no κ-Souslin tree. This extends V.1.3(viii), which says that, in ZFC, an inaccessible cardinal κ is weakly compact iff there is no κ-Aronszajn tree. We shall also show that under $V = L$, V.1.3(ii) may be extended.

We shall require the following characterisation of weak compactness, which is really just a $V = L$ analogue of Π_1^1-indescribability (V.1.3(iv)).

1.1 Lemma. *Assume $V = L$. Let κ be an inaccessible cardinal. Then κ is weakly compact iff, whenever $\varphi(\mathring{U}, \mathring{A}_1, \ldots, \mathring{A}_n)$ is a sentence of the language $\mathscr{L}(U, A_1, \ldots, A_n)$, if $A_1, \ldots, A_n \subseteq J_\kappa$ are such that*

$$(\forall U \subseteq J_\kappa)[\langle J_\kappa, \in, U, A_1, \ldots, A_n \rangle \models \varphi],$$

then for some $\alpha < \kappa$,

$$(\forall U \subseteq J_\alpha)[\langle J_\alpha, \in, U, A_1 \cap J_\alpha, \ldots, A_n \cap J_\alpha \rangle \models \varphi]. \quad \square$$

There are various ways of proving 1.1. One way is to make minor modifications to the proof that Π_1^1-indescribability characterises weak compactness in ZFC (V.1.3(iv)). Another way is to prove that under the assumption $V = L$, the