Chapter V
The Story of $0^{\#}$

In this chapter we investigate the effect upon $V = L$ of the postulated existence of various large cardinals in the universe. This represents a different approach to constructibility from that adopted hitherto. Previously we have been looking at the *internal* structure of the constructible universe. We now step back and regard *L from the outside* as it were.

It is assumed that the reader has a prior acquaintance with large cardinal theory. Admittedly, our account is self-contained (except for the omission of some proofs); but the results we shall obtain cannot really be appreciated without some familiarity with the standard theory of the cardinal properties concerned. The relevant material can be found in *Drake* (1974) and *Jech* (1978).

We shall make considerable use of model-theoretic techniques, usually for models of the languages $\mathscr{L}(A_1, \ldots, A_n)$. It will be convenient to use some of the standard notation of model theory. In particular, we shall write the satisfaction relation as

$$\langle M,\in, A_1, \ldots, A_n\rangle \models \varphi$$

rather than

$$\models_{\langle M,\in, A_1, \ldots, A_n\rangle} \varphi .$$

We shall also not bother to distinguish between an element, x, of a structure and the constant, $\mathring{x}$, of $\mathscr{L}_V$ which denotes it. If $t(\mathring{x}_0, \ldots, \mathring{x}_m)$ is a term of $\mathscr{L}_M(A_1, \ldots, A_n)$ (so $x_0, \ldots, x_m \in M$), we write $t^{\mathscr{A}}(x_0, \ldots, x_m)$ for the interpretation of the term $t(\mathring{x}_0, \ldots, \mathring{x}_m)$ in the structure $\mathscr{A} = \langle M, \;, A_1, \ldots, A_n\rangle$.

We shall also speak of models of ZFC, BS, etc. In each such case we mean these theories formulated in the language $\mathscr{L}$, and not in LST as was originally the case.

1. A Brief Review of Large Cardinals

A cardinal κ is said to be *weakly inaccessible* iff it is an uncountable, regular limit cardinal, and (*strongly*) *inaccessible* iff it is uncountable and regular and has the property that $(\forall \lambda < \kappa)(2^\lambda < \kappa)$. It is clear that all inaccessible cardinals are weakly inaccessible, and that if the GCH be assumed then the two notions of inaccessibility coincide.