# *Chapter IV*
# $\kappa^+$-Trees in $L$ and the Fine Structure Theory

In this chapter we shall investigate natural generalisations of the Souslin and Kurepa hypotheses to cardinals above $\omega_1$. In the case of the Souslin hypothesis this will require some combinatorial properties of $L$ which we shall only be able to prove by developing the theory of the constructible hierarchy more thoroughly than hitherto. (This is the so-called "fine-structure theory".)

## *1. $\kappa^+$-Trees*

Let $\kappa$ be an infinite cardinal. The concept of a *$\kappa$-tree* was defined in Chapter III. By a *$\kappa$-Aronszajn tree* we mean a $\kappa$-tree with no $\kappa$-branch. A *$\kappa$-Souslin tree* is a $\kappa$-tree with no antichain of cardinality $\kappa$. Just as in III.1.2, every $\kappa$-Souslin tree is $\kappa$-Aronszajn. And by arguments as in III.1.3, if $\kappa$ is regular, then any $(\kappa, \kappa)$-tree with unique limits which has no $\kappa$-branch has a subtree which is $\kappa$-Aronszajn; if in addition the original tree has no antichain of cardinality $\kappa$, it has a subtree which is $\kappa$-Souslin. The regularity of $\kappa$ is essential here. Indeed, for singular $\kappa$, the notion of a $\kappa$-tree is somewhat pathalogical. For example, if $\kappa$ is singular there is a $(\kappa, \kappa)$-tree with no $\kappa$-branch and no antichain of cardinality $\kappa$ (namely the disjoint union of the well-ordered sets $(\kappa_\nu, \varepsilon)$, $\nu < \mathrm{cf}(\kappa)$, where $(\kappa_\nu \mid \nu < \mathrm{cf}(\kappa))$ is cofinal in $\kappa$), but every $\kappa$-tree has an antichain of cardinality $\kappa$ (an easy exercise). We therefore restrict our attention to $\kappa$-trees for regular $\kappa$ only. Since we shall be assuming $V = L$ for our main results, GCH will hold, and hence the only regular limit cardinals are the (strongly) inaccessible cardinals. In this context we may therefore expect the notion of a $\kappa$-tree for $\kappa$ a regular limit cardinal to be bound up with the notion of large cardinals. As we shall see in Chapter VII, this is in fact the case. In this chapter we concentrate only upon the successor cardinals.

By a *$\kappa^+$-Kurepa tree* we mean a $\kappa^+$-tree with $\kappa^{++}$ many $\kappa^+$-branches. (Adopting a similar definition of a "$\kappa$-Kurepa tree" for inaccessible $\kappa$ does not lead to any interesting notions, as we see in Exercise 3. More care is required in order to define a reasonable notion of a $\kappa$-Kurepa tree in this case.) As in III.2.1, the existence of a $\kappa^+$-Kurepa tree can be shown to be equivalent to the existence of a certain kind of family of subsets of $\kappa^+$. Moreover, the proof that such a family exists in $L$ is a straightforward generalisation of the proof for the $\omega_1$ case, given in III.2.2.