Chapter III
ω_1-Trees in L

Tree theory forms a rich and interesting part of combinatorial set theory, having applications in other parts of set theory as well as in other areas of mathematics (in particular, in general topology). We study trees here because tree theory is greatly enhanced by the assumption $V = L$, and affords a good example of the application of the methods of constructibility theory. In this chapter we concentrate on ω_1-trees, and as we shall demonstrate, these arise out of some very basic questions in mathematics. Later chapters deal with generalisations to higher cardinals.

1. The Souslin Problem. ω_1-Trees. Aronszajn Trees

The Souslin Problem has its origin in a classical theorem of Cantor concerning the real line. In order to consider this theorem we need some definitions.

A *densely ordered set* is a linearly ordered set $\langle X, \leqslant \rangle$ such that whenever $x, y \in X$ and $x < y$, there is a $z \in X$ such that $x < z < y$.

An *interval* in a linearly ordered set $\langle X, \leqslant \rangle$ is a subset of X of the form

$$(x, y) = \{z \in X \mid x < z < y\}$$

for some $x, y \in X$, $x < y$. (We call this set the interval *determined by* x and y.)

An *ordered continuum* is a densely ordered set $\langle X, \leqslant \rangle$ such that whenever Y is a subset of an interval of X, there is a least $z \in X$ such that $(\forall y \in Y)(y \leqslant z)$ and a greatest $x \in X$ such that $(\forall y \in Y)(x \leqslant y)$. (We call z the *supremum* of Y, x the *infimum* of Y.)

A linearly ordered set is said to be *open* if it has no end-points.

A subset Y of a densely ordered set $\langle X, \leqslant \rangle$ is said to be *dense* in X if, whenever $x, z \in X$ are such that $x < z$, there is a $y \in Y$ such that $x < y < z$.

Cantor proved that, considered as a linearly ordered set, the real line ($\mathbb{R}$) is characterised, up to isomorphism, by being an open, ordered continuum having a countable dense subset (the rationals). In 1920, M. Souslin asked whether a natural weakening of these conditions still suffices to characterise $\mathbb{R}$.