Chapter II
The Constructible Universe

In Zermelo-Fraenkel set theory, the notion of what constitutes a set is not really defined, but rather is taken as a basic concept. The Zermelo-Fraenkel axioms describe the properties of sets and the set-theoretic universe. For instance, if \( X \) is an infinite set, the Power Set Axiom tells us that there is a set, \( \mathcal{P}(X) \), which consists of all subsets of \( X \). But the other axioms do not tell us very much about the members of \( \mathcal{P}(X) \), or give any indication as to how big a set this is. The Axiom of Comprehension says that \( \mathcal{P}(X) \) will contain all sets which are describable in a certain, well-defined sense, and AC will provide various choice sets and well-orderings. But the word "all" in the phrase "all subsets of \( X \)" is not really explained. Of course, as mathematicians we are (are we not?) quite happy with the notion of \( \mathcal{P}(X) \), and so long as there are no problems, Zermelo-Fraenkel set theory can be taken as a perfectly reasonable theory. But as we know, ZFC set theory does have a major drawback: there are several easily formulated questions which cannot be answered on the basis of the ZFC axioms alone. A classic example is the status of the continuum hypothesis, \( 2^\omega = \omega_1 \). It can be argued that this cannot be decided in ZFC because the ZFC axioms do not say just what constitutes a subset of \( \omega \); hence we cannot relate the size of \( \mathcal{P}(\omega) \) to the infinite cardinal numbers \( \omega_\alpha, \alpha \in \text{On} \). (The formal proof of the undecidability of CH is rather different from the above "plausibility argument".)

One way of overcoming the difficulty of undecidable questions is to extend the theory ZFC, to obtain a richer theory which provides more information about sets. (An alternative solution is simply to accept as a fact of life that some questions have no answer.) One highly successful extension of ZFC is the constructible set theory of Gödel. In this theory the notion of a "set" is made precise (at least relative to the ordinals). The idea is as follows.

The fundamental picture of the set-theoretic universe which the Zermelo-Fraenkel axioms supply is embodied in the cumulative hierarchy of sets. We commence with the null set, \( \emptyset \), and obtain all other sets by iteratively applying the (undescribed) power set operation, \( \mathcal{P} \). Thus:

\[
V_0 = \emptyset; \quad V_{\alpha+1} = \mathcal{P}(V_\alpha); \quad V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha, \quad \text{if } \text{lim}(\lambda).
\]

Then

\[
V = \bigcup_{\alpha \in \text{On}} V_\alpha.
\]