Chapter II
The Constructible Universe

In Zermelo-Fraenkel set theory, the notion of what consitutes a set is not really defined, but rather is taken as a basic concept. The Zermelo-Fraenkel axioms describe the *properties* of sets and the set-theoretic universe. For instance, if X is an infinite set, the Power Set Axiom tells us that there is a set, $\mathscr{P}(X)$, which consists of *all* subsets of X. But the other axioms do not tell us very much about the members of $\mathscr{P}(X)$, or give any indication as to how big a set this is. The Axiom of Comprehension says that $\mathscr{P}(X)$ will contain all sets which are *describable* in a certain, well-defined sense, and AC will provide various choice sets and well-orderings. But the word "all" in the phrase "all subsets of X" is not really explained. Of course, as mathematicians we are (are we not?) quite happy with the notion of $\mathscr{P}(X)$, and so long as there are no problems, Zermelo-Fraenkel set theory can be taken as a perfectly reasonable theory. But as we know, ZFC set theory does have a major drawback: there are several easily formulated questions which cannot be answered on the basis of the ZFC axioms alone. A classic example is the status of the *continuum hypothesis*, $2^\omega = \omega_1$. It can be argued that this cannot be decided in ZFC because the ZFC axioms do not say just what constitutes a subset of ω; hence we cannot relate the size of $\mathscr{P}(\omega)$ to the infinite cardinal numbers ω_α, $\alpha \in \mathrm{On}$. (The formal *proof* of the undecidability of CH is rather different from the above "plausibility argument".)

One way of overcoming the difficulty of undecidable questions is to extend the theory ZFC, to obtain a richer theory which provides more information about sets. (An alternative solution is simply to accept as a fact of life that some questions have no answer.) One highly successful extension of ZFC is the *constructible set theory* of Gödel. In this theory the notion of a "set" is made precise (at least relative to the ordinals). The idea is as follows.

The fundamental picture of the set-theoretic universe which the Zermelo-Fraenkel axioms supply is embodied in the cumulative hierarchy of sets. We commence with the null set, $\emptyset$, and obtain all other sets by iteratively applying the (undescribed) power set operation, $\mathscr{P}$. Thus:

$$V_0 = \emptyset; \qquad V_{\alpha+1} = \mathscr{P}(V_\alpha); \qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \qquad \text{if } \lim(\lambda).$$

Then

$$V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha .$$