Chapter VI

Inductive Definitions

"Let X be the smallest set containing ... and closed under ---." A definition expressed in this form is called an inductive definition. We have used this method of definition repeatedly in the previous chapters; for example, in defining the notions of Δ_0 formula, Σ formula, infinitary formula, provable using the $\mathfrak{M}$-rule, etc. In this chapter we turn method into object by studying inductive definitions in their own right. We will see that their frequent appearance is more than an accident.

1. Inductive Definitions as Monotonic Operators

Let A be an arbitrary set. An n-ary *inductive definition* on A is simply a mapping Γ from n-ary relations on A to n-ary relations on A which is *monotone increasing*; i.e. for all n-ary relations R, S on A

$$R \subseteq S \quad \text{implies} \quad \Gamma(R) \subseteq \Gamma(S).$$

If $\Gamma(R)=R$ then R is a *fixed point of* Γ.

1.1 Theorem. *Every inductive definition on A has a smallest fixed point. Indeed, there is a relation R such that:*

(i) $\Gamma(R)=R$,

(ii) *for any relation S on A, if $\Gamma(S) \subseteq S$ then $R \subseteq S$.*

Proof. Let $\mathbf{C}=\{S \subseteq A^n \mid \Gamma(S) \subseteq S\}$. Since $A^n \in \mathbf{C}$, $\mathbf{C}$ is non-empty. Let $R=\bigcap \mathbf{C}$. Since (ii) now holds by definition it remains to prove (i), that is, that $\Gamma(R)=R$. Let S be an arbitrary member of $\mathbf{C}$. Since $R \subseteq S$ and Γ is monotone we have $\Gamma(R) \subseteq \Gamma(S)$, but $\Gamma(S) \subseteq S$, so $\Gamma(R) \subseteq S$. Since S was an arbitrary member of $\mathbf{C}$, and $R=\bigcap \mathbf{C}$, we have $\Gamma(R) \subseteq R$. To show that $R \subseteq \Gamma(R)$ it suffices to prove that $\Gamma(R) \in \mathbf{C}$. But since $\Gamma(R) \subseteq R$ we have, by monotonicity, $\Gamma(\Gamma(R)) \subseteq \Gamma(R)$ so $\Gamma(R) \in \mathbf{C}$. ▯

The proof of 1.1, while correct, tells us next to nothing about the smallest fixed point of Γ and is certainly not the way we mentally justify a typical inductive definition. Let us look at an example.