Chapter IV

Elementary Results on $\mathbb{HYP}_{\mathfrak{M}}$

We have seen, in Chapter III, how admissible sets provide a tool for the study of infinitary logic by giving rise to those countable fragments which are especially well-behaved. In this chapter we begin the study of $\mathbb{HYP}_{\mathfrak{M}}$ by means of the logical tools developed in Chapter III.

1. On Set Existence

Given $\mathfrak{M}$ we form the universe of sets $\mathbb{V}_{\mathfrak{M}}$ on $\mathfrak{M}$ and speak glibly about arbitrary sets $a \in \mathbb{V}_{\mathfrak{M}}$. In practice, however, one seldom considers the impalpable sets of extremely high rank. There is even a feeling that these sets have a weaker claim to existence than the sets one normally encounters. Without becoming too philosophical, we want to touch here on the question: If we assume $\mathfrak{M}$ as given, to the existence of what sets are we more or less firmly committed?

$\mathbb{HYP}_{\mathfrak{M}}$ is the intersection of all models $\mathfrak{A}_{\mathfrak{M}}$ of KPU$^+$ and is an admissible set above $\mathfrak{M}$. There appears to be a certain *ad hoc* feature to $\mathbb{HYP}_{\mathfrak{M}}$, however, since it might depend on the exact axioms of KPU$^+$ in a sensitive way. You would expect that if you took a stronger theory than KPU$^+$ (say throw in Power, or Infinity or Full Separation) that more sets from $\mathbb{V}_{\mathfrak{M}}$ would occur in all models of this stronger theory. That, for $\mathfrak{M}$ countable, this cannot happen, lends considerable weight to the contension that $\mathbb{HYP}_{\mathfrak{M}}$ is here to stay.

Of the two results which follow, the second implies the first. We present them in the opposite order for expository and historical reasons.

A set $S \subseteq \mathfrak{M}$ is *internal* for $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ if there is an $a \in A$ such that $S = a_E = \{x \in \mathfrak{A}_{\mathfrak{M}} \mid xEa\}$.

1.1 Theorem. *Let* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ *be a countable structure for* L. *Let T be a consistent theory (finitary or infinitary) which is* Σ_1 *on* $\mathbb{HYP}_{\mathfrak{M}}$ *and which has a model of the form* $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$. *Let* $S \subseteq M$ *be such that S is internal for every such model of T. Then* $S \in \mathbb{HYP}_{\mathfrak{M}}$.