Chapter XX

Abstract Embedding Relations

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Abstract model theory deals with the generalization of the concept of a logic. A logic consists of a family of objects called formulas, a family of objects called structures and a binary relation between them, called satisfaction. Various properties of logics, however, can be phrased without direct reference to the formulas, but rather, by considering as the basic concept the class of structures which are the models of some (complete) theory. The previous two chapters have given plenty of evidence for this. In Section XVIII.3 we studied amalgamation properties and in Chapter XIX, the Robinson property, both of which fit this approach. In Chapter XIX we even went a step further: we looked into the possibility of axiomatizing abstract equivalence relations between structures, such as they arise naturally from logics in the form of $\mathscr{L}$-equivalence. There we studied the question under which circumstances such an equivalence relation does indeed come from a logic $\mathscr{L}$.

Algebra, on the other hand, deals with classification of algebraic structures and their extensions. The paradigm of algebraic classification theory, and, for that matter, the paradigm of model-theoretic classification theory, is Steinitz' theory of fields and their algebraic and transcendental extensions. But many of the examples studied in algebra, such as locally finite groups or Banach spaces, are not fit for first-order axiomatizations. Though classes of algebras can be axiomatized, if necessary, with the help of generalized quantifiers, this approach does not necessarily help us to axiomatize the corresponding notion of extensions.

In this chapter we axiomatize the notion of $\mathscr{L}$-extensions, but, contrary to the approach in Chapter XIX, we are not that much interested in the case where it is derived from a logic $\mathscr{L}$. We are rather interested in the question: Under which conditions can certain constructions and proofs from model theory be carried out in a framework which resembles more that of universal algebra or algebra in general?

Very often, axiomatizations grow out of a better understanding of proofs. First, they serve only to structure and clarify the flow of reasoning, but sometimes they gain their own significance and reach maturity. If this happens, new branches of mathematical activity emerge.

Examples from history are the emergence of Hilbert and Banach spaces; universal algebra and model theory of first-order logic, abstract model theory, and here especially, the framework of abstract classes. The abstract classes have their