Chapter XIX
Abstract Equivalence Relations

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For a regular logic $\mathscr{L}$, let $\sim\ =\ \equiv_{\mathscr{L}}$ be the equivalence relation obtained by saying that two structures are $\sim$-equivalent iff they satisfy the same sentences of $\mathscr{L}$. The isomorphism relation $\cong$ is automatically a refinement of $\sim$—that is, isomorphic structures are $\sim$-equivalent— $\sim$ itself is a refinement of elementary equivalence $\equiv$, and $\sim$ is preserved under both renaming and reduct. This last property simply means that upon renaming, or taking reducts of $\sim$-equivalent structures, we obtain $\sim$-equivalent structures. Furthermore, if $\mathscr{L}[\tau]$ is a set for every vocabulary τ, then the collection of equivalence classes given by $\sim$ on $\mathrm{Str}(\tau)$ has a cardinality. (Briefly, we say that $\sim$ is *bounded*). This paper is mainly concerned with abstract equivalence relations $\sim$ on $\bigcup_\tau \mathrm{Str}(\tau)$, having the above-mentioned properties as well as the *Robinson property* so that for every $\mathfrak{M}$, $\mathfrak{N}$, and τ with $\tau = \tau_{\mathfrak{M}} \cap \tau_{\mathfrak{N}}$,

$$\text{if}\quad \mathfrak{M} \upharpoonright \tau \sim \mathfrak{N} \upharpoonright \tau \quad \text{then for some } \mathfrak{A},$$
$$\mathfrak{M} \sim \mathfrak{A} \upharpoonright \tau_{\mathfrak{M}} \quad \text{and} \quad \mathfrak{N} \sim \mathfrak{A} \upharpoonright \tau_{\mathfrak{N}}.$$

If $\sim\ =\ \equiv_{\mathscr{L}}$, then $\sim$ has the Robinson property iff $\mathscr{L}$ satisfies the Robinson consistency theorem. If, in addition, $\mathscr{L}[\tau]$ is a set for all τ, and if all sentences in $\mathscr{L}$ have a finite vocabulary, then the Robinson consistency theorem holds in $\mathscr{L}$ iff $\mathscr{L}$ is compact and has the interpolation property (see Corollary 1.4). Every bounded equivalence relation $\sim$ with the Robinson property satisfies the equation $\sim\ =\ \equiv_{\mathscr{L}}$ for *at most one* logic $\mathscr{L}$ (see Corollary 3.4). This result can be extended to equivalence relations corresponding to compact logics (see Theorem 3.11). Moreover, we have that $\sim\ =\ \equiv_{\mathscr{L}}$ for *exactly one* logic $\mathscr{L}$ iff $\sim$ is *separable* by quantifiers, in the sense that whenever $\mathfrak{M}$ and $\mathfrak{N}$ are not $\sim$-equivalent, there is a quantifier Q such that $\sim$ is a refinement of $\equiv_{\mathscr{L}(Q)}$ and $\mathfrak{M} \not\equiv_{\mathscr{L}(Q)} \mathfrak{N}$ (see (ii) of Theorem 3.10). Even if $\sim$ is not separable by quantifiers, there is still a strongest logic $\mathscr{L}$ such that $\sim$ refines $\equiv_{\mathscr{L}}$. This $\mathscr{L}$ is compact and can be written as $\mathscr{L} = \mathscr{L}\{Q \mid \sim$ is a refinement of $\equiv_{\mathscr{L}(Q)}\}$ (see Corollary 3.3 and (i) of Theorem 3.10).

The Robinson property of $\mathscr{L}$ can also be coupled with such properties as $[\omega]$-incompactness. Then $\equiv_{\mathscr{L}}$ will coincide with $\cong$ below the first uncountable measurable cardinal μ_0 (see Theorem 1.7), and the infinitary logic $\mathscr{L}_{\mu_0\omega}$ can be interpreted in $\mathscr{L}$ in some natural sense (refer to Theorem 1.12).

Some of the results in Section 1 can be extended to logics for enriched structures, such as topological, uniform, and proximity structures, as discussed in Section 2.