Chapter XVIII
Compactness, Embeddings and Definability

by J. A. Makowsky

This chapter presents an overview of the author's joint work with S. Shelah in abstract model theory, which had started as early as 1972. It is mainly based on our papers (Makowsky–Shelah–Stavi [1976]; Makowsky–Shelah [1979, 1981, 1983]) and on an unpublished manuscript of S. Shelah (Shelah [198?e]) which he wrote while this chapter came into being. The present exposition, however, tries to give a more coherent picture by putting all our results into a single perspective together with results of M. Magidor, H. Mannila, D. Mundici, and J. Stavi.

The main theme of this chapter is abstract model theory proper, especially the relationship between various compactness, embedding, and definability properties which do not characterize first-order logic. More precisely, we look at various classes of logics defined axiomatically, such as compact logics, logics satisfying certain model existence or definability properties. The classes of logics are sometimes further specified by set-theoretic parameters, such as finitely generated, absolute, set presentable, bounds on the size function, or by set-theoretic assumptions such as large cardinal axioms. Within such classes of logics we want to explore which other properties of logics follow from the axiomatic description of the class. In Chapter III first-order logic was characterized in this way. In Chapter XVII the class of absolute logics was studied. Most of the other chapters (with the exception of Chapters XIX and XX) study families of logics which bear some inherent similarity which stems from the way they evolved, such as infinitary logics or logics based on cardinality quantifiers, and establish particular model-theoretic results for those logics. In this chapter we want to clarify the conceptual and metalogical relationship between these model theoretic properties. Success in this program can be achieved in three ways: by establishing non-trivial connections between these properties; by applying the former to gain new insight about particular logics previously studied; and by using this insight to construct new examples of logics, and ultimately, by showing, that our list of examples is, in some reasonable sense, exhaustive.

The chapter consists of four sections, in each of which one aspect of abstract model theory is developed to a certain depth.

Section 1 is devoted to compactness properties and is almost self-contained. Its main results are the abstract compactness theorem and the description of the compactness spectrum. Here a thorough understanding of various compactness