Chapter XVI

Borel Structures and Measure and Category Logics

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Two very significant ways in which the theory of models has been extended beyond first-order logic are the enrichment of the syntax to include additional quantifiers and the restriction of the class of structures to be considered. These two means will be brought together in this chapter. The focus here will be on the model theory of structures whose domain and some subset of whose definable relations and functions can be built from the subsets of $\mathbb{R}^n$ that are most frequently encountered in analysis and topology: the Borel sets (see Section 1.1 for precise definitions). Such first-order structures are studied in Sections 1.2 and 1.3. The most widely-used notions of size for Borel sets are category, measure, and uncountability. The model theory of "Borel structures," when the syntax is expanded to allow quantifiers capable of expressing one or more of these concepts will be explored in the final two sections of the chapter.

Friedman initiated the study of the structures and logics which are the subject of this chapter in the series of abstracts (Friedman [1978], [1979a] and [1979b]). Most of the major results presented here are due to him. Friedman has expressed the hope that by restricting the available class of structures for a theory to those which are in some sense Borel, the negative results obtained by using arbitrary uncountable or non-separable structures can be largely eliminated. That is to say, the abundance of positive results found in many areas of mathematical practice for countable, separable, or even well-behaved uncountable and non-separable structures may also be discovered for the classes of structures to be discussed here.

At present it is not at all clear that the study of these structures and logics can quite realize the aims sketched above. Nevertheless, the techniques and notions that have already been developed seem powerful, and the wealth of interesting problems that arise in this area surely warrants our further attention.

1. Borel Model Theory

1.1. Measure and Category Logic, and Borel Structures

In this chapter, all theories will be built from a countable vocabulary τ, even though we will usually suppress explicit reference to τ. First-order logic and several finitary extensions of it obtained by adjoining various combinations of the new