Chapter XV
Topological Model Theory

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1. Topological Structures

A (one-sorted) topological structure $\tilde{\mathfrak{A}} = (\mathfrak{A}, \alpha)$ with vocabulary τ consists of a τ-structure $\mathfrak{A}$ and a topology α on A. Familiar examples are topological spaces ($\tau = \varnothing$), and topological groups and fields. Note that in general we do not assume that the relations and operations of $\mathfrak{A}$ are compatible with α. This in contrast to Robinson [1974].

A *logic* for topological structures is a pair $(\mathscr{L}, \models)$, where $\mathscr{L}[\tau]$ is a class (of "$\mathscr{L}$-sentences") for each vocabulary τ and $\models$ is a relation between topological structures and $\mathscr{L}$-sentences. We will now assume that the axioms of a regular logic hold for topological structures (see Examples 1.1.1 and Discussion 1.2). The relativization axiom is, of course, an exception to this general assumption. The reader should consult Section 2 for a description of the many-sorted case.

1.1. Three Logics for Topological Structures

We first consider *quantification over α and the logic $\mathscr{L}^t_{\text{mon}}$*. We say that an $\mathscr{L}^t_{\text{mon}}[\tau]$-formula is built up from atomic $\mathscr{L}_{\omega\omega}[\tau]$-formulas and atomic formulas

$$t \in X,$$

where t is a τ-term and X a "set variable" (which ranges over α), using $\neg$, $\wedge$, $\vee$, $\forall x$, $\exists x$, $\forall X$, $\exists X$. The semantics are self-explanatory. A logic (for $\tau = \varnothing$) equivalent to $\mathscr{L}^t_{\text{mon}}$ was introduced in Grzegorczyk [1951] and Henson *et al.* [1977].

1.1.1 Examples. (i) $(A, \alpha) \models \forall X\, \forall Y(\exists x\, \exists y(x \in X \wedge y \in Y) \rightarrow \exists x((x \in X \wedge x \in Y) \vee (\neg x \in X \wedge \neg x \in Y)))$ or, more briefly, $(A, \alpha) \models \forall X, Y(X \neq \varnothing \wedge Y \neq \varnothing \rightarrow (X \cap Y \neq \varnothing \vee X \cup Y \neq \text{universe}))$ which holds iff (A, α) is connected.

(ii) $(A, F, \alpha) \models \forall X\, \exists Y\; Y = f^{-1}(X)$ iff $F\colon A \rightarrow A$ is continuous with respect to α.

(iii) $(A, B, \alpha) \models \exists X\, \forall x(P(x) \leftrightarrow x \in X)$ iff B is open, i.e., $B \in \alpha$.