Part E

Logics of Topology and Analysis

This part of the book is devoted to logics which presuppose different kinds of structures than those underlying first-order logic and its extensions so far dealt with in Parts B, C and D.

Chapter XIV is about logics where the underlying structure is a probability space, a structure with a countably additive probability measure. In addition to the usual propositional operations, the basic form of quantification is given by allowing formulas

$$(Px \geq r)\phi(x),$$

which means that the probability of the set $\{x: \phi(x)\}$ is at least r. Structures take the form of probability spaces with countably additive measures. To have a successful theory here a number of changes in perspective must be made. In the first place, one must arrange things so that all definable sets are measurable. As a result, the logics considered here are not closed under the usual quantifiers $\forall$ and $\exists$. Consequently, these logics do not contain first-order logic, nor do they satisfy all the assumptions on logics given in the general definition. They also have model-theoretic properties that have no first-order analogue, like the Law of Large Numbers.

While the lack of ordinary quantifiers entail a loss in expressive power, we can compensate for that, in part, by the use of countable conjunctions and disjunctions, as in $\mathscr{L}_{\omega_1\omega}$, since such operations preserve measurability (due to countable additivity of probability measures). Expressed in terms of admissible sets, one finds the appropriate forms of completeness and compactness results. Interestingly, there is also an analogue of the Robinson consistency property, which fails for $\mathscr{L}_{\omega_1\omega}$. This chapter should be read after reading the relevant sections of Chapter VIII.

In his retiring address as president of the Association for Symbolic Logic in 1972, Abraham Robinson (Robinson [1973]) asked what logic for topological structures was the analogue of first-order logic for algebraic structures. Chapter XV presents the work that has gone into this problem. Obviously the structures to be considered are of the form $(\mathfrak{M}, \tau)$, where τ is a topology on the domain of the first-order structure $\mathfrak{M}$. Examples include topological space, topological groups, and topological fields. It has taken a lot of effort to arrive at what appears to be