

# Part D

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## *Second-Order Logic*

This part of the book is devoted to the study of second-order logic and some of its applications. We discuss the two chapters in the opposite order from that in which they appear.

Chapter XIII is about monadic second-order logic, logic that allows quantification over arbitrary subsets of the domain, but not over arbitrary relations or functions. While this does not make any difference on structures like the natural numbers with plus and times, where sequences can be coded by numbers, it turns out to make an enormous difference in more algebraic settings. In these cases, monadic second-order logic is a good source of theories that are both highly expressive yet manageable. Section 2 illustrates the uses of finite automata and games in the proof of decidability results. It begins with a simple case, the monadic theory of finite chains, which it works out in complete detail, and shows how the method generalizes to a number of results, including one of the most famous, Rabin's theorem on the decidability of the monadic second-order theory of two successor functions. In Section 3 more model-theoretic methods, generalized products, are used to prove some of the same and related results. Some undecidability results are also presented. Proofs of these have to be novel, since we are dealing with theories where one cannot interpret first-order arithmetic.

If we think of monadic second-order logic as the part of second-order logic obtained restricting the quantification in a simple definable manner, we can ask whether there are any other natural sublogics that can be obtained by restricting the second-order quantifiers in some other first-order definable manner. There is one other. Namely, one might quantify not over arbitrary functions, but over permutations of the domain. This is called permutational logic. It arose in Shelah's study of symmetric groups. However, as it turns out, that's all! Up to a strong form of equivalence, the only sublogics of second-order logic given by first-order restricted second-order quantifiers are first-order logic, monadic second-order logic, permutational logic, and full second-order logic. This result, first proved in Shelah [1973c], is established by some new methods in Chapter XII. In addition, a number of newer, related results are presented.