Chapter XI
Applications to Algebra

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In contrast to the situation in first-order finitary logic, the applications of infinitary logic to algebra are so scattered throughout the literature that it is extremely difficult to discern any coherent pattern. Nevertheless, there are some interesting applications; and, in this chapter, we will survey a few of them. This survey will primarily be for the benefit of the non-specialist. That being so, proofs will not always be given in detail, since our aim is simply to present enough background to state a result, indicate its significance, and explain how infinitary logic enters into the statement of the result and/or its proof.

The separate sections are organized by algebraic subject matter and are essentially independent of each other. The first four sections involve $\mathscr{L}_{\infty\omega}$, while the fifth and sixth make use of $\mathscr{L}_{\infty\kappa}$ for arbitrary κ. The last section is simply a collection of references to other relevant literature.

The first two sections of our survey deal with applications of logic to algebra in the purest sense that results expressible in algebraic terms are proved by logical means. The first section's concern—arguably the most important application to date of infinitary model theory to algebra—is the construction by Macintyre and Shelah of non-isomorphic universal locally finite groups of the same cardinality. In the second section we examine the use by Baldwin of some profound results in the model theory of $\mathscr{L}_{\omega_1\omega}$ to count the number of subdirectly irreducible algebras in a variety. The remaining sections involve applications in which logical notions are employed in the expression as well as in the proof of a result so as to provide new insight into an algebraic notion or problem.

Sections 3, 4 and 5 make use of the notion of infinitary equivalence. In Section 3, the back-and-forth characterization of $\mathscr{L}_{\infty\omega}$ equivalence is used to formulate precisely and prove the heuristic principle in algebraic geometry known as Lefschetz's principle. Classification theorems in abelian group theory are studied in Section 4 to see what information can be gained from their proofs about the $\mathscr{L}_{\lambda\omega}$-equivalence of abelian groups. Section 5 gives a characterization of the algebras in a variety which are $\mathscr{L}_{\infty\kappa}$-equivalent to a free algebra, and the question of the existence of non-free such algebras is studied, in general, and specifically in the variety of abelian groups. Finally, Section 6 presents both Hodges' formalization of the notion of a concrete (or effective) construction and an examination of his use of it in proving that certain algebraic constructions are not concrete.

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