Chapter X
Game Quantification

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Game quantification interacts with the model theory of infinitary logics, abstract model theory, generalized recursion theory, and descriptive set theory. The aim of this chapter is to examine these connections and give some applications of the game quantifiers to the above areas of mathematical logic.

The chapter is divided into four sections. The first presents the basic notions and the interpretation of infinite strings of quantifiers via two-person infinite games. Section 2 deals with the interaction between game quantification and global definability theory, the main theme being that certain second-order statements can be reduced to formulas involving the game quantifiers which can, in turn, be approximated by formulas of $L_{\infty\omega}$. This section also includes a proof of Vaught's covering theorem, as well as applications of game quantification to the model theory of $L_{\omega_1\omega}$ and admissible fragments. In Section 3, we show that the game logics are absolute and unbounded, and most of the model-theoretic properties of these logics will then follow from this fact. Section 4, the final section, discusses the interaction with local definability theory. Here we consider the basic relation of the game quantifiers to inductive definability and higher recursion theory, and give some of their uses in descriptive set theory.

1. Infinite Strings of Quantifiers

This section presents the main definitions and basic results about infinite strings of quantifiers $(Q_0x_0Q_1x_1Q_2x_2\ldots)$ where, for each $i = 0, 1, 2, \ldots, Q_i$ is the *existential quantifier* $\exists$ or the *universal quantifier* $\forall$ on a set A. The interpretation of such strings is via two-person infinite games of perfect information. We first describe the interpretation in an informal way and indicate the expressive power of certain infinite strings. The precise definitions involve the notions of a winning strategy and a winning quasistrategy. The Gale–Stewart theorem is then proven and used to push negation through infinite strings in certain cases.

Throughout this section, A is a non-empty infinite set, $A^{<\omega} = \bigcup_{n\in\omega} A^n$ is the set of all finite sequences from A, and A^ω is the collection of all infinite sequences of elements of A. We use variables $x, y, z, \ldots$ to denote elements of A, variables