Chapter IX

Larger Infinitary Languages

by M. A. DICKMANN

1. The Infinitary Languages $\mathscr{L}_{\kappa\lambda}$ and $\mathscr{L}_{\infty\lambda}$

The motivations underlying the study of infinitary languages which are given in the introduction to Chapter VIII will also serve well here, thereby relieving us of the need to make further comments.

Recall that for infinite cardinals κ, λ, with $\kappa \geq \lambda$, the language $\mathscr{L}_{\kappa\lambda}$ is constructed by prescribing a stock of individual variables of cardinality κ and a list τ of finitary non-logical symbols called the *vocabulary*. Furthermore, $\mathscr{L}_{\kappa\lambda}$ contains connectives and quantifiers permitting the formation of:

  (i) the negation of any expression;
 (ii) conjunctions and disjunctions of any number (strictly) fewer than κ expressions;
(iii) existential and universal quantifications over any set of fewer than λ variables.

The formal definition of the set of *expressions* of $\mathscr{L}_{\kappa\lambda}$ is left as an exercise. *Formulas* will be expressions containing less than λ free variables. This restriction is made in order to provide the means for "quantifying out" all free variables in a formula.

The class-language $\mathscr{L}_{\infty\lambda}$ will have as its formulas those formulas of all the languages $\mathscr{L}_{\kappa\lambda}$, for $\kappa \geq \lambda$ (with the same vocabulary); that is, $\mathscr{L}_{\infty\lambda}$ allows conjunctions and disjunctions of any *set* of its *formulas* but permits quantifications only over fewer than λ variables. The language $\mathscr{L}_{\infty\infty}$ contains as formulas those formulas of the languages $\mathscr{L}_{\infty\lambda}$ for all infinite cardinals λ.

The semantics of $\mathscr{L}_{\kappa\lambda}$, $\mathscr{L}_{\infty\lambda}$ and $\mathscr{L}_{\infty\infty}$ are defined by straightforward extrapolation of the first-order definition of satisfaction, for instance, by declaring that $\bigwedge_{i\in I} \phi_i$ is true iff each ϕ_i is true, etc..

In the remainder of this section, we will present a number of examples illustrating the use and the expressive power of the languages we have just introduced. They were chosen so as to provide a foretaste of what general results we may or may not expect from the model theory of these larger infinitary languages. Indeed, some of the model-theoretic results in Section 3 are elaborations on some of the examples which follow.