Part II. Compactness Regained

5. *Admissibility*

In passing from $\mathscr{L}_{\omega\omega}$ to $\mathscr{L}_{\infty\omega}$ a very substantial gain in expressive power is achieved. As is to be expected, however, there is a considerable price to pay. Many of the very useful properties of $\mathscr{L}_{\omega\omega}$—most notably compactness—are no longer enjoyed by $\mathscr{L}_{\infty\omega}$. If we restrict our attention to $\mathscr{L}_{\omega_1\omega}$, then some of these properties are salvaged. For example, interpolation, and a reasonable form of completeness can be thus regained. Compactness, however, clearly still fails. To obtain an omitting types result, we considered countable fragments L_B of $\mathscr{L}_{\omega_1\omega}$. Though completeness looks even better in this framework, interpolation, for example, fails. Thus, while on the one hand we want to deal with parts of $\mathscr{L}_{\omega_1\omega}$ small enough to be manageable, on the other hand, we would nevertheless like them to be large enough to be closed, for example, under finding interpolants. For this latter consideration, it would be preferable if the pieces that we deal with were given in some absolute way, since then, using them to give bounds would be more meaningful from "the first-order" point of view. $L_{\omega_1\omega}$ itself, as a fragment of $\mathscr{L}_{\infty\omega}$, is given by cardinality conditions, and so is certainly not "first-order".

In order to introduce the notion that has proven fruitful in this respect, we will assume, first of all—without doing any of this explicitly—that the syntax and semantics of $\mathscr{L}_{\infty\omega}$ are given within set theory. That is, we assume that sentences are sets, structures are sets, satisfaction is a ternary relation between structures, formulas, and functions from variables, *etc.* For any transitive set B we will thus be able to define $L_B = L_{\infty\omega} \cap B$; that is, the formulas of L_B are those formulas of $L_{\infty\omega}$ in B. Mild assumptions on B will guarantee that L_B is a fragment in the sense we have been using. Somewhat stronger conditions will give us a great deal of closure, and, when combined with countability, will even give a form of compactness.

5.1. KP *and Admissible Sets*

An *admissible set* is a transitive set A, such that $\langle A, \in\rangle$ is a model of a certain theory KP, the initials standing for Kripke and Platek. Kripke [1964a, b] and Platek [1966] were engaged in trying to generalize recursion theory to the ordinals. They were following the earlier work of Takeuti [1960], [1965] and Tugué [1964] who were studying recursion on the set of all ordinals, and Kreisel–Sacks [1965] whose metarecursion theory, in turn, followed from earlier work of Kleene [1955b] on recursive ordinals and hyperarithmetic sets. For a more complete history, the reader should consult the introduction to Barwise [1975].

In order to present the theory KP, we must first recall the Lévy hierarchy of formulas of a language containing the binary relation symbol $\in$ and perhaps other symbols as defined in Lévy [1965]. The collection of Δ_0-*formulas* is the smallest collection of formulas containing the atomic formulas, closed under the boolean connectives of $\neg$, & and $\vee$, and under bounded quantification. (That is, if φ is a