Chapter VII
Decidability and Quantifier-Elimination

By A. Baudisch, D. Seese, P. Tuschik, and M. Weese

The decidability of the elementary theory for a given class K of structures reflects a certain low expressive power of the elementary language with respect to that class. Therefore, it is natural to look for stronger logics L such that K has a decidable L-theory. The rigorous establishment of decidability for the L-theory of K often provides results about the L-definable properties and L-equivalence of structures in K. This means, then, that investigations into the decidability of the L-theory of K are closely related to the L-model theory of K.

In this chapter we will investigate the decidability of such logics. We will concentrate on Malitz quantifiers (particularly on cardinality quantifiers) and Härtig quantifiers as well as on stationary logic. The first result in this direction was the decidability of the theory of unary predicates without equality in the logic with the quantifier "there are $\aleph_\alpha$ many". This result was proven in a fundamental paper by Mostowski [1957]. Topological and monadic second-order logics are treated in other chapters of this volume; and, we therefore, will not consider them here. However, we wish to emphasize at this point that results concerning the latter do have important consequences for the material that will be presented in our discussions.

Our chapter is basically organized along the lines sketched below. First, with respect to three main methods of proving decidability, there is a division into three sections which are respectively entitled *Quantifier-Elimination*, *Interpretations*, and *Dense Systems*. In each of these the general method is introduced and then clarified with respect to several concrete classes of structures. These classes are: the class of modules and abelian groups (Section 1), the class of well-orderings (Section 2), and the classes of linear orderings and boolean algebras (Section 3). At the end of each subsection we refer to some further results without making any claims that the discussions given present a complete picture of the material. However, the reader will find references to most of the corresponding investigations in the bibliography given at the end of the volume.

Much of the material of this chapter is related to our text (see Baudisch–Seese–Tuschik–Weese [1980]), in which the reader can find more detailed proofs as well as some similar investigations on the class of trees.

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