Chapter VI

Other Quantifiers: An Overview

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Generalized quantifiers were introduced by Mostowski [1957] as a means of generating new logics. In the meantime, their study has greatly developed, so that today there are more quantifiers in the literature than there are abstract model theorists under the sun. In any logic $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^i)_{i \in I}$ one does not need to introduce specific formation rules for renaming and substitution; for, upon adding to the finite set of logical symbols of $\mathscr{L}_{\omega\omega}$ one new symbol for each Q^i, all sentences in $\mathscr{L}$ are obtainable by an induction procedure on strings of symbols, pretty much as in $\mathscr{L}_{\omega\omega}$. One can gödelize sentences and start studying the axiomatizability and decidability of theories in $\mathscr{L}$. One might even go as far as to write down the proof of a theorem in $\mathscr{L}$ and then have it published in some mathematical journal. For infinitary logics this all seems to be a bit more problematic.

There are several ways to introduce quantifiers. For instance, nonlinear prefixes of existentially and universally quantified variables may be regarded as quantifiers as is discussed in Section 1. Quantifiers are also used for transforming concepts such as isomorphism, well-order, cardinality, continuity, metric completeness, and the "almost all" notion into primitive logical notions such as $=$ (see Sections 2 and 3).

There is no reason why quantifiers introduced via the above definability criteria should also preserve the nice algebraic properties of $\mathscr{L}_{\omega\omega}$. Indeed, in many cases they do not. However, in a final section of this chapter we will briefly describe a novel approach to quantifiers, an approach that is based on the fact that every separable Robinson equivalence relation $\sim$ on structures is canonically representable as $\mathscr{L}$-equivalence, $\equiv_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}_{\omega\omega}\{Q \mid \equiv_{\mathscr{L}(Q)}$ is coarser than $\sim\}$. In addition to this, $\mathscr{L}$ turns out to have compactness and interpolation: The open, interior quantifiers and their n-dimensional variants can be introduced in this way, starting from a suitable approximation of homeomorphism.

We do not aim at an encyclopedic coverage here. Rather, we only aim to present an anthology of the most significant facts and techniques in the variegated realm of quantifiers. In line with this, highly developed quantifiers or special topics are discussed in detail in Chapters IV, V, VII, and XV.

Throughout this chapter $\mathscr{L}^{\mathrm{mII}}$ will be taken to mean second-order logic with universal and existential quantifiers over *unary* relations. Moreover, we will also write $\mathscr{L}(Q^i)_{i \in I}$ instead of $\mathscr{L}_{\omega\omega}(Q^i)_{i \in I}$.