Chapter V

Transfer Theorems and Their Applications to Logics

by J. H. SCHMERL

This chapter is primarily concerned with the general problem of transferring results about one logic, say $\mathscr{L}(Q_1)$, to another logic, say $\mathscr{L}(Q_\alpha)$. A typical such property is $\aleph_0$-compactness. It is known from Chapter IV that $\mathscr{L}(Q_1)$ is $\aleph_0$-compact. Under certain set-theoretic assumptions on α discussed in this chapter, the logic $\mathscr{L}(Q_1)$ transfers to $\mathscr{L}(Q_\alpha)$. In such cases we can then conclude that $\mathscr{L}(Q_\alpha)$ is also $\aleph_0$-compact. The logics that we consider in this chapter are variants and generalizations of $\mathscr{L}(Q_1)$, and the properties of these logics which we are most concerned with are compactness and recursive enumerability for validity.

1. The Notions of Transfer and Reduction

After presenting the basic definitions that allow useful model-theoretic comparisons between logics, we present applications to compactness and recursive enumerability of logics and to two-cardinal questions.

1.1. Transfer

The substantive theme of this chapter is the notion of transfer and we will begin our explorations with

1.1.1 Definition. Suppose $\mathscr{L}_0$ and $\mathscr{L}_1$ are two logics which have exactly the same syntax but differ in their semantics. Then $\mathscr{L}_0$ *transfers to* $\mathscr{L}_1$ iff every sentence which is satisfiable relative to $\mathscr{L}_0$ is also satisfiable relative to $\mathscr{L}_1$. In symbols, we write $\mathscr{L}_0 \to \mathscr{L}_1$.

Transfer becomes quite fruitful when there is *mutual transfer*, when both $\mathscr{L}_0 \to \mathscr{L}_1$ and $\mathscr{L}_1 \to \mathscr{L}_0$ hold. For, in this situation $\mathscr{L}_0$ and $\mathscr{L}_1$ have exactly the same valid sentences, so that a syntactic property known to hold for $\mathscr{L}_0$ will also hold for $\mathscr{L}_1$. For example, if $\mathscr{L}_0$ has the Beth property, then so does $\mathscr{L}_1$. In this chapter we will generally be concerned with two properties which are especially amenable to verification using the methods of transfer. These properties are