Part B

Finitary Languages with Additional Quantifiers

Part B of the book is devoted to the study of logics with added quantifiers and the applications of such. The logics considered, for the most part, express properties of ordinary structures. Logics with additional quantifiers based on richer structures are studied in Part E.

Chapter IV begins the discussion by investigating the logic $\mathscr{L}(Q_1)$ with the quantifier "there exist uncountably many." It also discusses various extensions of $\mathscr{L}(Q_1)$ including stationary logic $\mathscr{L}(\mathrm{aa})$ and the Magidor–Malitz logic $\mathscr{L}^{<\omega}$. The primary emphasis of the chapter is on the method of constructing models of size $\aleph_1$ used by Keisler [1970] to prove his completeness theorem for $\mathscr{L}(Q_1)$, a method that has become one of the standard tools of the subject. Each of these logics comes with its own intended concepts of "small" set and "large" set. The basic idea of Keisler-type proofs is to use an elementary chain $\langle A_\alpha : \alpha < \omega_1 \rangle$ of countable non-standard or "weak" models to build a standard model, one where the quantifier has its intended interpretation. The key step is always from A_α to $A_{\alpha+1}$, constructing $A_{\alpha+1}$ so that all small definable subsets of A_α stay fixed, but where a fixed definable subset of A_α that is supposed to be large receives a new element.

Chapter V discusses the general problem of transferring results known about $\mathscr{L}(Q_\alpha)$ to some other $\mathscr{L}(Q_\beta)$, especially the problem of taking results known about $\mathscr{L}(Q_1)$, where we have powerful techniques for building models, to $\mathscr{L}(Q_{\beta+1})$ for larger β. For example, if we assume the Generalized Continuum Hypothesis, it follows that the axioms and rules that are complete for $\mathscr{L}(Q_1)$ are also complete for any logic of the form $\mathscr{L}(Q_{\beta+1})$, as long as $\aleph_\beta$ is regular. In general, this chapter depends heavily on various set-theoretical assumptions which are independent of the usual axioms of set theory, however.

Chapter VI surveys and compares the strength of a host of other logics with additional quantifiers. One of these is the class of partially ordered quantifiers like Q^{H} whose meaning is given by: $Q^{\mathrm{H}}x, y; z, w\phi(x, y, z, w)$ is true just in case for every x there is a y, and for every z there is a w, such that y depends only on x, w only on z, such that $\phi(x, y, z, w)$. Quantifiers of this kind are called partially ordered because they are often written:

$$\begin{matrix} \forall x\, \exists y \\ \forall z\, \exists w \end{matrix}\ \phi(x, y, z, w).$$