Chapter III
Characterizing Logics

by J. Flum

The model theory of first-order logic is well developed. It provides general results and methods which enable us to study and classify the models of systems of first-order axioms. Among these general results of wide applicability are the completeness theorem, the compactness theorem, and the Löwenheim–Skolem theorem. Thus, for example, the completeness theorem leads to decidability results; in many cases we obtain for a given system of axioms models with special properties using a compactness argument; finally the Löwenheim–Skolem theorem tells us that we can restrict to countable structures when classifying—with respect to its first-order properties—models of a system of axioms.

Much effort was spent in finding languages which strengthen the first-order language and which are

(i) sufficiently strong to allow the formulation of interesting systems of axioms and properties of structures which are not expressible in first-order logic, and

(ii) still simple enough to yield general principles and results which are useful in investigating and classifying models.

Taking into account the situation for first-order logic, it is not surprising that many logicians attempted to find logics satisfying the analogues of the completeness, the compactness, and the Löwenheim–Skolem theorems. That this search could not be successful was shown by the following two results, both of which are due to Lindström [1969]:

(1) First-order logic is a maximal logic with respect to expressive power satisfying the compactness theorem and the Löwenheim–Skolem theorem.

(2) First-order logic is a maximal logic satisfying the completeness theorem and the Löwenheim–Skolem theorem.

Let us point out some consequences of these results.

(a) They tell us that first-order logic is a natural logic, if one accepts the completeness (or the compactness) and the Löwenheim–Skolem property as natural properties. I suspect that most mathematicians do not accept the Löwenheim–Skolem property as natural. Quite the contrary, as Wang [1974, p. 154] remarked: "When we are interested in set theory or classical analysis, the Löwenheim theorem is usually taken as a sort of defect (often thought to be inevitable) of the first-order logic. Therefore, what is established (by Lindström's theorems)