Chapter VIII
Recursion on Ordinals

In the preceding two chapters we have put our attention on the generalized recursion theories which arise from ordinary recursion theory by the introduction of functionals of types 2 and higher. As we saw in § VI.7, these theories can equally well be viewed as theories of effective computability over domains which include such functionals. In this chapter, we shall study theories of effective computability over (initial segments of) the class of ordinal numbers.

The intuitive notion of effective computability for functions from ordinals to ordinals is again based on an idealized computing machine M which is equipped to store and manipulate ordinals according to a given program and produce an ordinal as answer. Many features of the "language" in which the programs for M are to be written are unchanged from earlier notions of computability, but there are two new basic instructions. These ensure that if H is a partial computable functions, then so are F and G defined by:

$$F(\rho, \boldsymbol{\mu}) \simeq \sup^{+}_{\pi<\rho} H(\pi, \boldsymbol{\mu}),$$

and

$$G(\boldsymbol{\mu}) \simeq \text{"least" } \pi \,.\, H(\pi, \boldsymbol{\mu}) \simeq 0.$$

The justification for including these schemes lies in a generalization of the notion of finiteness, which we call *metafiniteness.*

In ordinary recursion theory an object is finite iff it is in a one-to-one correspondence with a natural number — that is, with an element of the fundamental domain. If this were the only property of finite objects considered, we might say that an object is metafinite iff it is in a one-to-one correspondence with an ordinal. Of course, given the Axiom of Choice, this would make every set metafinite! Instead, we observe that in addition, every finite set of natural numbers is computable and indeed is in a *computable* one-to-one correspondence with a natural number. Hence we shall call an object *metafinite* iff it is in a computable one-to-one relationship with an ordinal.

Our basic principle of intuitive calculability over the ordinals is that a metafinite sequence of computations may be regarded as a completed totality and an answer drawn from the sequence of results of these computations. Thus,