Chapter V
Δ_2^1 and Beyond

Most of the analysis of the first level of the analytical hierarchy in Chapter IV rests on the representation of Π_1^1 sets in terms of well-orderings (Theorem IV.1.1), and for many years after these results were known there seemed to be no hope of extending any of the methods or results to higher levels. Since W is a Π_1^1 set it cannot be used directly to represent all Σ_2^1 or Π_2^1 relations, and no analogue of W at higher levels was apparent.

In § 1 we formulate the abstract pre-wellordering property and show that much of the structure of Π_1^1 and Δ_1^1 relations is due solely to the fact that Π_1^1 has this property. Furthermore, it is easily seen that Σ_2^1 also has the pre-wellordering property and this leads to the conclusion that a strong analogy exists between Π_1^1 and Σ_2^1. This correspondence will be reinforced in § VIII.3 where we discuss two generalizations of recursion theory for which Π_1^1 and Σ_2^1 are exactly the classes of "semi-recursive" relations.

The pre-wellordering property cannot be proved for other classes in the analytical hierarchy without further set-theoretical hypotheses beyond ZFC. In §§ 2 and 3 we discuss two such hypotheses — the hypothesis of constructibility ($V = L$) and the hypothesis of projective determinacy (PD). The principal results are (1) if $V = L$, then Σ_r^1 has the pre-wellordering property for all $r \geqslant 2$, whereas (2) if PD, then the classes which have the pre-wellordering property are $\Pi_1^1, \Sigma_2^1, \Pi_3^1, \Sigma_4^1, \Pi_5^1, \ldots$. These hypotheses also imply analogues of many of the results of §§ IV.5–7 for higher levels of the analytical and projective hierarchies.

We turn then to extensions of the results of §§ IV.3–4, which might be termed the study of Δ_1^1 and $\boldsymbol{\Delta}_1^1$ "from below". Here the results are mainly negative: no analogue of the Borel hierarchy suffices to exhaust any of the classes $\boldsymbol{\Delta}_r^1$ for $r \geqslant 2$, and similarly for the effective hierarchies and Δ_r^1. On the other hand, the classes of sets which comprise these analogues are themselves somewhat similar in structure to the class of (effective) Borel sets. The classical (boldface) versions lead to significant extensions of the reuslts of § IV.5, while the effective versions will be seen in §§ VI.5–6 to be closely connected with certain generalized recursion theories. Finally in § 6 we consider some facts peculiar to Δ_2^1 which lead to a hierarchy for the Δ_2^1 relations on numbers.