# *Chapter III*
# Hierarchies and Definability

In the preceding chapter we saw that the semi-recursive relations are exactly those which arise from the recursive relations by existential number quantification (II.4.12). In this chapter we study the relations which arise from the recursive relations by all kinds of quantification: existential, universal, number, and function. After classifying these relations according to the number and type of quantifiers used and establishing the simplest combinatorial properties of this classification in §§ 1 and 2, we relate it to other notions of definability. In § 3 we compare the complexity of definition of an inductive operator $\Gamma$ with that of the set $\bar{\Gamma}$. In § 4, we investigate the relationship between the complexity of a subset A of ${}^{\omega}\omega$ and that of its elements. In § 5 we show that the relations we are considering are exactly those definable in certain natural first- and second-order formal languages. Finally in § 6 we introduce the method of forcing to extend and complete some earlier results.

## *1. The Arithmetical Hierarchy*

**1.1 Definition.** The class of *arithmetical relations* is the smallest class of relations containing the recursive relations and closed under number quantification ($\exists^0$ and $\forall^0$).

We next define a classification of the arithmetical relations based on the number of quantifiers needed to define a relation.

**1.2 Definition** (The Arithmetical Hierarchy). For all $r$,

(i) $\Sigma_0^0 = \Pi_0^0 =$ the class of recursive relations;

(ii) $\Sigma_{r+1}^0 = \{\exists^0\mathsf{P}\colon \mathsf{P} \in \Pi_r^0\}$;

(iii) $\Pi_{r+1}^0 = \{\forall^0\mathsf{P}\colon \mathsf{P} \in \Sigma_r^0\}$;

(iv) $\Delta_r^0 = \Sigma_r^0 \cap \Pi_r^0$;

(v) $\Delta_{(\omega)}^0 = \bigcup \{\Sigma_r^0 \cup \Pi_r^0\colon r \in \omega\}$.

It is immediate by induction on $r$ that all of the classes $\Sigma_r^0$ and $\Pi_r^0$, and hence $\Delta_{(\omega)}^0$, are included in the class of arithmetical relations. The converse inclusion is immediate from Theorem 1.5 below.