Chapter 8

Set Recursion and Higher Types

Primitive recursive set functions and rudimentary set functions have been around for some time, see Jensen-Karp [72], Gandy [40], and also Devlin [19] for their basic properties and further references. The step to a general recursion theory on sets came rather late. There is, perhaps, a reason why: Primitive recursive and rudimentary set functions were introduced to elucidate the rather restricted recursion-theoretic nature of the constructible hierarchy, and in the hands of Jensen [71] have become an important tool in the fine structure theory of L.

Full set recursion was introduced by D. Normann [124], and later rediscovered by Y. Moschovakis, as a tool for developing a companion theory for Kleene-recursion in higher types. The theory has, however, a wider scope and we shall present a general version in the first part of this chapter. In this we follow the exposition in Normann [124]. The approach of Moschovakis uses inductive definability, but the end result is substantially the same.

In Section 3 we work out the detailed connection with Kleene-recursion in higher types. Some of this work has its origin in the theses of Harrington [53], MacQueen [98], and Normann [122]. We believe that the general set-theoretic approach adds both simplicity and insight.

As a testing ground for this belief we turn in Section 4 to the degree theory in higher types. We present a fairly simple priority argument involving ${}^3E$, allowing the reader to explore the full intricacies of the general theory for him- or herself. We just want to make the point that set recursion is a very natural computation theory to use in the study of degrees of functionals.

8.1 Basic Definitions

Set recursion in a relation R on the universe of sets V is generated by the schemes for the functions rudimentary in R augmented with the diagonalization scheme.

8.1.1 Definition. Let $R \subseteq V$ be a relation. The class of partial functions *set-recursive relative to* R is inductively defined by the following clauses

(i) $f(x_1, \ldots, x_n) = x_i$ $\qquad e = \langle 1, n, i \rangle$

(ii) $f(x_1, \ldots, x_n) = x_i - x_j$ $\qquad e = \langle 2, n, i, j \rangle$