Chapter 6

Degree Structure

The study of degrees, in particular of r.e. degrees, is a characteristic and important part of recursion theory. And no account of general recursion theory can be claimed to be successful unless at least an introduction to notions of *reducibilities* and the associated *degree theory* is given. This is precisely what we will do in this chapter: to present an *introduction* to this topic within the general framework of infinite theories and to give an example of a non-trivial result in the extended framework.

But we should really like to do something more. In the spirit of an axiomatic analysis we want to determine the "true" domain for degree theory and priority arguments. This is the question we turn to in Section 6.3. Our discussion is fragmentary and we do not claim any complete solution. Indeed there may not be any well-defined "solution". But we hope that this section may give some clue as to how far recursion-theoretic regularities extend.

Our discussion is, in principle, self-contained, but some familiarity with the basic notions of α-recursion theory would be helpful: we recommend the introductory paper of R. A. Shore, *α-recursion theory* [152], in a precise sense we continue his discussion in this chapter.

6.1 Basic Notions

The setting is an infinite computation theory Θ on a prewellordered domain $(\mathfrak{A}, \preccurlyeq)$, see Definition 5.1.5. We shall need a suitable notion of enumeration and of parametrization of the Θ-semicomputable sets. But as usual we must preface our definitions by introducing some necessary notation.

Let f be a mapping which to every $x \in A$ gives us a canonical Θ-index for a Θ-finite set, i.e. $f(x)$ is an index for the function $\mathbf{E}_{W^x}$, where W^x is the Θ-finite set associated with x. It will be convenient to write the mapping as

$$f = \lambda x \cdot W^x,$$

but we should always remember that the value of f at x is a canonical Θ-index for the Θ-finite set W^x.