Appendix A
Coding into Structures and Theories

Several of our applications of local degree theory have relied on our ability to code certain information into structures and theories. Thus we needed to associate a lattice with each degree, and we accomplish this by fixing a set of a given degree and *coding* it into a lattice in such a way that the set can be recovered recursively from any presentation of the lattice. Also, one of the methods which we use to prove undecidability results is to code one theory T_0 into another theory T_1. We accomplish this by describing a recursive translation which takes any sentence θ_0 of the language for T_0 into a sentence θ_1 in the language for T_1 so that $\theta_0 \in T_0 \Leftrightarrow \theta_1 \in T_1$. The undecidability of T_1 will then imply the undecidability of T_0. The major theories in which we have an interest are the theories of true first and second order arithmetic, the theory of (distributive) lattices, and the theory of graphs.

1. Degrees of Presentations of Lattices

Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a countable lattice. A *presentation* of $\mathscr{L}$ is an isomorphic copy $\mathscr{P} = \langle P, \leqslant_P, \vee_P, \wedge_P \rangle$ of $\mathscr{L}$ such that P is a recursive set. The *degree* of $\mathscr{P}$ is then the join of the degrees of $\leqslant_P$, $\vee_P$ and $\wedge_P$.

We wish to prove a result used in Chap. VIII.2 which assigns a lattice $\mathscr{L}_{\mathbf{a}}$ to each degree $\mathbf{a}$. Thus given $\mathbf{a} \in \mathbf{D}$, we choose a set A of degree $\mathbf{a}$ and code A into a lattice $\mathscr{L}_{\mathbf{a}}$. We show that $\mathscr{L}_{\mathbf{a}}$ has a presentation of degree $\mathbf{a}$, and that A can be recovered recursively from any presentation of $\mathscr{L}_{\mathbf{a}}$.

1.1 Theorem. *For any degree* $\mathbf{a}$, *there is a countable lattice* $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ *such that*:

(i) $\langle L, \leqslant \rangle$ *has a presentation of degree* $\mathbf{a}$.
(ii) *Any presentation of* $\langle L, \leqslant \rangle$ *has degree* $\geqslant \mathbf{a}$.

Proof. For each $n \in N$, let $\mathscr{L}_n$ be the lattice of Fig. 1.1. Thus $\mathscr{L}_n$ is the lattice with $2n + 9$ elements, its universe is $L_n = \{d^n, e^n, c_0^n, c_1^n, a_1^n, \ldots, a_{n+2}^n, b_0^n, \ldots, b_{n+2}^n\}$ which is ordered by specifying that exactly the following relations hold: