Chapter XI

Bounding Minimal Degrees with Recursively Enumerable Degrees

The constructions of minimal degrees which have been presented to this point have been oracle constructions. Most of the theorems in Part C, however, were originally proved using *full approximation* constructions. Although different constructions have different features, the common thread in full approximation constructions is that both the set of minimal degree and the trees on which this set lies are simultaneously constructed through recursive approximations.

In this chapter, we prove that every non-zero recursively enumerable degree bounds a minimal degree. The proof we give involves a full approximation construction.

1. Trees Permitted by Recursively Enumerable Sets

Let C be a non-recursive, recursively enumerable set, and let $h: N \to N$ be a one-one recursive function enumerating C. Let $C^s = \{h(x): x \leqslant s\}$. We construct a set $A \leqslant_T C$ by recursive approximation. Thus we define a recursive sequence of strings $\{\alpha_s: s \in N\}$ and let $A(x) = \lim_s \alpha_s(x)$ for all $x \in N$. (Recall that in oracle constructions, A was more simply defined by $A = \cup\{\alpha_s: s \in N\}$.) C will control the recursive approximation $\{\alpha_s: s \in N\}$ by subjecting the construction of this approximation to the following constraint:

(1) $\qquad \forall x, s \in N(C^s \restriction x = C \restriction x \to \alpha_s \restriction x \subset A)$.

Condition (1) will guarantee that $A \leqslant_T C$, as is shown in the following lemma.

1.1 Yates Permitting Lemma. *Let* $\{\alpha_s: s \in N\}$ *be a recursive sequence of elements of* $\mathscr{S}_2$. *Define* $A \subseteq N$ *by* $A(x) = \lim_s \alpha_s(x)$ *for all* $x \in N$. *Assume that* (1) *holds. Then* $A \leqslant_T C$.

Proof. To compute $A(x)$, search for the least $s \in N$ such that $C^s \restriction x + 1 = C \restriction x + 1$ and $\text{lh}(\alpha_s) > x$. Since h is a one-one recursive function which enumerates C, such an s can be found through the use of a C oracle. By (1), $A(x) = \alpha_s(x)$. $\square$