Chapter VIII
Countable Usls

The countable ideals of $\mathscr{D}$ are characterized in this chapter. In particular, we show that if $\mathscr{L}$ is a countable usl with least element, then $\mathscr{L} \hookrightarrow^* \mathscr{D}$. This characterization of countable ideals of $\mathscr{D}$ is applied to answer questions about automorphisms, elementary equivalence, and definability over $\mathscr{D}$. Results proved in Appendix A and Appendix B.3 are used in this chapter.

1. Countable Ideals of $\mathscr{D}$

We show that every countable usl with least element is isomorphic to an ideal of $\mathscr{D}$. We first state the theorem which provides the necessary table, and then introduce the new kinds of trees needed for the construction. We conclude by characterizing the countable ideals of $\mathscr{D}$. Notation and definitions are carried over from Chap. VII.

1.1 Definition. Let $\{\mathscr{L}_i : i \in N\}$ be given such that for each $i \in N$, $\mathscr{L}_i = \langle L_i, \leqslant_i, \vee_i \rangle$ is a usl and $\mathscr{L}_0 \subseteq \mathscr{L}_1 \subseteq \cdots$. We define the usl $\mathscr{L} = \bigcup\{\mathscr{L}_i : i \in N\} = \langle L, \leqslant, \vee \rangle$ by letting $L = \bigcup\{L_i : i \in N\}$, defining $a \leqslant b$ for $a, b \in L$ if for some $i \in N$, $a, b \in L_i$ and $a \leqslant_i b$, and defining $a \vee b$ for $a, b \in L$ to be the element c such that $c = a \vee_i b$ where i is the least element of N such that $a, b \in L_i$.

If each L_i in Definition 1.1 is finite and has a least element, then each $\mathscr{L}_i$ is a lattice, since every finite usl with least element is a lattice.

Let $\mathscr{L} = \langle L, \leqslant, \vee \rangle$ be a usl, and let $\mathscr{L}_i = \langle L_i, \leqslant_i, \vee_i \rangle$ be a finite usl such that $\mathscr{L}_i \subseteq \mathscr{L}$. Let $a \in L - L_i$ be given, and let $\mathscr{L}^* = \langle L^*, \leqslant^*, \vee^* \rangle$ be the smallest usl such that $L \cup \{a\} \subseteq L^*$ and $\mathscr{L}^* \subseteq \mathscr{L}$. Then $\mathscr{L}^*$ is finite since each element b of L^* is expressible as $b = \bigvee\{d : d \in M\}$ for some $M \subseteq L \cup \{a\}$. Thus we note the following fact:

1.2 Remark. Let $\mathscr{L} = \langle L, \leqslant, \vee \rangle$ be a countable usl with least and greatest elements u_0 and u_1 respectively. Then there is a sequence $\{\mathscr{L}_i : i \in N\}$ of finite lattices such that for each $i \in N$, $\mathscr{L}_i = \langle L_i, \leqslant_i, \vee_i \rangle$, $L_i = \{u_0, \ldots, u_{f(i)}\}$, u_0, $u_1 \in L_0$, and

(i) $\quad \forall i \in N(\mathscr{L}_i \subseteq \mathscr{L}_{i+1}) \quad$ (as a usl).

(ii) $\quad \mathscr{L} = \bigcup\{\mathscr{L}_i : i \in N\}$.