Chapter VII
Finite Lattices

We completely characterize the finite ideals of $\mathscr{D}$ in this chapter as the set of all finite lattices. It is not known whether all finite lattices have finite homogeneous lattice tables, so we replace these tables with *weakly homogeneous sequential lattice tables* which are possessed by all finite lattices. We extend the methods of Chap. VI, using such tables to embed finite lattices as ideals of $\mathscr{D}$. This embedding theorem is used to locate decidable fragments of Th($\mathscr{D}$); the $\forall_2$-theory of $\mathscr{D}$ is decidable, but the $\forall_3$-theory of $\mathscr{D}$ is undecidable. Results from Appendices A.2 and B.2 are used in this chapter.

1. Weakly Homogeneous Sequential Lattice Tables

We define the tables needed to characterize the finite ideals of $\mathscr{D}$, motivating the definition by discussing the way in which the properties specified by the tables relate to the proofs of various lemmas in Chap. VI. Throughout this chapter, f will denote a non-decreasing recursive function such that $f(x) \geqslant 2$ for all $x \in N$. Recall that for such an f, $\mathscr{S}_f = \{\sigma \in \mathscr{S}\colon \forall x \in N(\sigma(x) < f(x))\}$. Our forcing conditions will be f-branching trees, i.e., trees $T\colon \mathscr{S}_f \to \mathscr{S}_f$, and will be referred to as trees, dropping the word *f-branching*.

The key lattice theoretic fact used in Chapter VI to embed finite distributive lattices as ideals of $\mathscr{D}$ was that all such lattices have finite homogeneous lattice tables. Since it is unknown whether all finite lattices have finite homogeneous lattice tables, we use a weaker type of table to embed all finite lattices as ideals of $\mathscr{D}$. All finite lattices have countable lattice tables, so we will have to find conditions to replace finiteness and homogeneity for the tables. Lemma VI.1.4, which asserts that the coding obtained from the table produces a usl homomorphism, will then have the same proof in this new setting.

It is crucial to the proof of a computation lemma that each tree T used in the construction have the property that for all strings $\sigma \in \mathrm{dom}(T)$, $S_\sigma = \{n \in N\colon T(\sigma * n)\downarrow\}$ is finite. This set S_σ however, must be generated from a lattice table which will not be finite, in order to always be able to find interpolants for the greatest lower bound preservation property. This apparent conflict is resolved by building the table as the union of an increasing sequence of finite usl tables, with the