Chapter VI
Finite Distributive Lattices

We continue our study of the finite ideals of $\mathscr{D}$ in this chapter by showing that every finite distributive lattice is isomorphic to an ideal of $\mathscr{D}$. This result is proved using techniques extending those introduced in Chap. V. Different trees are used, and we introduce *tables* which provide reduction procedures from the top degree of the ideal; these tables are obtained from representations of distributive lattices. As an application, we show that the set of minimal degrees forms an automorphism base for $\mathscr{D}$.

Many of the applications which we obtain in later chapters from the complete characterization of the countable ideals of $\mathscr{D}$ can be obtained from the fact that all countable distributive lattices are isomorphic to ideals of $\mathscr{D}$. We use Exercise 4.17 of this chapter to indicate how to obtain the characterization of distributive ideals of $\mathscr{D}$. This exercise allows the reader to proceed directly to Chap. VIII.2 from the end of this chapter.

The results of Appendix B.1 are needed for this chapter.

1. Usl Representations

Tables built from lattice representations provide the starting point for defining the trees used in this chapter. We begin to motivate the use of such tables in this section. Recall that $\mathscr{S}_p$ is the set of all strings of integers $< p$.

The trees used in this chapter are p-branching trees, i.e. trees $T: \mathscr{S}_p \to \mathscr{S}_p$. We will refer to p-branching trees as *trees* during this section, dropping the words *p-branching.*

Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a finite lattice, with $L = \{u_0, \ldots, u_n\}$. We assume, without loss of generality, that for all $i, j \leqslant n$, if $i < j$ then $u_j \nleqslant u_i$ and u_0 and u_n are, respectively, the least and greatest elements of L. We wish to construct a function $g_n: N \to N$ such that $\mathbf{D_L} = \langle \mathbf{D[0, g_n]}, \leqslant \rangle$ is a lattice which is isomorphic to $\mathscr{L}$ under the map $\psi: L \to \mathbf{D_L}$ given by $\psi(u_i) = \mathbf{g_i}$ for all $i \leqslant n$, where $\mathbf{D[0, g_n]} = \{\mathbf{g_0}, \ldots, \mathbf{g_n}\}$. As in the construction of a minimal degree, we will define a sequence of trees $\{T_i : i \in N\}$ and choose $g_n \in \bigcap\{T_i : i \in N\}$.

For all $i \leqslant n$ and $z \in N$, we can view $g_i \restriction z$ as a string. Thus, for example, if $\sigma = 2 * 0 * 3$, we write $\sigma \subset g_i$ if $g_i(0) = 2$, $g_i(1) = 0$ and $g_i(2) = 3$. We will