Chapter IV
High/Low Hierarchies

Hierarchies on both **D** and the arithmetical degrees are introduced. Properties of sets which lie in certain classes of this hierarchy are examined, and results obtained are used to find automorphism bases for certain classes of degrees.

1. High/Low Hierarchies

Post's Theorem implies that the arithmetical hierarchy gives rise to a hierarchy $\{\mathbf{D_n}: n \in N\}$ for the arithmetical degrees, where $\mathbf{D_n} = \{\mathbf{d} \in \mathbf{D}: \mathbf{d} \leqslant \mathbf{0^{(n)}}\}$. While this hierarchy has its uses, it is not a very good hierarchy for studying the degrees. Two of its drawbacks are that there are degrees which are not classified by this hierarchy, and that the classes of the hierarchy have little to do with the properties of the degrees in the classes. We therefore introduce new hierarchies which are better suited to classifying properties of degrees. We prove basic facts about these hierarchies in this section, and study some classes of the hierarchies in subsequent sections.

1.1 Definition. Let $n \geqslant 0$ be given. Define $\mathbf{L_n}$, the class of *low$_n$ degrees* by $\mathbf{L_n} = \{\mathbf{d} \leqslant \mathbf{0'}: \mathbf{d^{(n)}} = \mathbf{0^{(n)}}\}$, and $\mathbf{H_n}$, the class of *high$_n$ degrees* by $\mathbf{H_n} = \{\mathbf{d} \leqslant \mathbf{0'}: \mathbf{d^{(n)}} = \mathbf{0^{(n+1)}}\}$. Define **I**, the class of *intermediate degrees* by $\mathbf{I} = \{\mathbf{d} \leqslant \mathbf{0'}: \forall n \in N(\mathbf{0^{(n)}} < \mathbf{d^{(n)}} < \mathbf{0^{(n+1)}})\}$. $\{\mathbf{H_n}: n \in N\} \cup \{\mathbf{L_n}: n \in N\} \cup \{\mathbf{I}\}$ is the set of classes of the *high/low* hierarchy.

The high/low hierarchy induces a partition of $\mathbf{D[0, 0']}$. The low$_n$ degrees are the degrees below $\mathbf{0'}$ whose nth jump is as small as possible, and the high$_n$ degrees are the degrees below $\mathbf{0'}$ whose nth jump is as large as possible. A similar hierarchy can be defined on $\mathbf{D[a, a']}$ for each $\mathbf{a} \in \mathbf{D}$.

1.2 Definition. Fix $\mathbf{a} \in \mathbf{D}$ and $n \in N$. Define $\mathbf{L_n(a)}$, the class of **a**-*low$_n$ degrees* by $\mathbf{L_n(a)} = \{\mathbf{d} \in \mathbf{D[a, a']}: \mathbf{d^{(n)}} = \mathbf{a^{(n)}}\}$, and $\mathbf{H_n(a)}$, the class of **a**-*high$_n$ degrees* by $\mathbf{H_n(a)} = \{\mathbf{d} \in \mathbf{D[a, a']}: \mathbf{d^{(n)}} = \mathbf{a^{(n+1)}}\}$. Define $\mathbf{I(a)}$, the class of **a**-*intermediate degrees* by $\mathbf{I(a)} = \{\mathbf{d} \in \mathbf{D[a, a']}: \forall n \in N(\mathbf{a^{(n)}} < \mathbf{d^{(n)}} < \mathbf{a^{(n+1)}})\}$. $\{\mathbf{L_n(a)}: n \in N\} \cup \{\mathbf{H_n(a)}: n \in N\} \cup \{\mathbf{I(a)}\}$ is the set of classes of the **a**-*high/low hierarchy*.

Figure 1.1 below gives a pictorial description of the **a**-high/low hierarchy for $\mathbf{a} \in \mathbf{D}$. The lower the degree in Fig. 1.1, the smaller it is in the ordering of **D**.