Chapter III
The Jump Operator

The jump operator is a naturally defined function taking each degree to a larger degree. It is also very closely related to the arithmetical hierarchy. We will study this relationship, as well as some algebraic structures whose universe is the set of degrees and on which the jump operator acts as a function.

The jump operator also allows us to pick out certain natural degrees other than **0**. Thus we are presented with certain natural intervals of degrees for which we can ask questions similar to those answered in Chap. II. We begin our study of *local degree theory*, i.e., the study of bounded intervals of degrees in this chapter. Many of the results proved throughout this book are local results which allow us to prove global theorems about the degrees.

1. The Arithmetical Hierarchy

The arithmetical hierarchy coincides with the $\exists_n/\forall_n$ hierarchy of sentences for the language of recursion theory specified below. It is introduced in this section, and characterizations of certain levels of this hierarchy are given.

There is one basic relation and one basic function for which we want symbols in our language for recursion theory. The first is the graph of the function φ of the Enumeration Theorem. The other is any one-one recursive correspondence $\pi: N^2 \to N$. π is called a *pairing function*. We will sometimes denote π by π_2, and note that for $n \geq 2$, we can recursively define $\pi_{n+1}: N^{n+1} \to N$ by

$$\pi_{n+1}(a_0, \ldots, a_n) = \pi(\pi_n(a_0, \ldots, a_{n-1}), a_n).$$

The underlying language for recursion theory is the language of the pure predicate calculus with equality, together with:

(1) For each $n \in N$, a constant symbol $\underline{n}$ whose interpretation is n.

(2) A five place relation symbol φ whose interpretation is the graph of the function of the Enumeration Theorem, i.e., $\{\langle \sigma, e, x, s, y \rangle: \varphi(\sigma, e, x, s)\!\downarrow = y\}$.

(3) A function symbol π of two places whose interpretation is the pairing function π.

Henceforth, we will identify each symbol in this language with its interpretation.