Chapter II
Embeddings and Extensions of Embeddings in the Degrees

We define the degrees of unsolvability in this chapter, and show that these degrees from an uppersemilattice. Much of the rest of this book will be devoted to studying this uppersemilattice. The study begins in this chapter, with sections on embedding theorems and on extensions of embeddings into the degrees. We also prove the decidability of a certain natural class of sentences about the degrees.

1. Uppersemilattice Structure for the Degrees

We are now ready to define the degrees of unsolvability, and to show that Turing reducibility induces a partial ordering on these degrees which gives rise to an uppersemilattice. In Section 4 we will prove that the degrees do not form a lattice.

We begin with some algebraic definitions.

1.1. Definition. A *partially ordered set* (*poset*) $\langle P, \leqslant \rangle$ is a set P together with a binary relation $\leqslant\ \subseteq P^2$ having the following properties:

(i) *Reflexivity*: $\forall x \in P(x \leqslant x)$.

(ii) *Antisymmetry*: $\forall x, y \in P(x \leqslant y \,\&\, y \leqslant x \to x = y)$.

(iii) *Transitivity*: $\forall x, y, z \in P(x \leqslant y \,\&\, y \leqslant z \to x \leqslant z)$.

1.2 Definition. An *uppersemilattice* (*usl*) is a triple $\langle P, \leqslant, \vee \rangle$ such that $\langle P, \leqslant \rangle$ is a poset, and $\vee: P^2 \to P$ (write $x \vee y = z$ for $\vee(x,y) = z$) satisfies:

$$\text{(i)} \qquad \forall x, y \in P(x \leqslant x \vee y \,\&\, y \leqslant x \vee y)$$

and

$$\text{(ii)} \qquad \forall x, y, u \in P(x \leqslant u \,\&\, y \leqslant u \to x \vee y \leqslant u).$$

Thus a usl is a poset in which every pair of elements has a least upper bound.

Clause (ii) of Definition 1.1 prevents the use of $\leqslant_T$ to directly transform N^N into a poset. This obstruction is circumvented by using certain equivalence classes of N^N, *the degrees*, as the domain of the poset. The equivalence relation used is the following.

1.3 Definition. For $f, g \in N^N$, define $f \equiv_T g$ if $f \leqslant_T g$ and $g \leqslant_T f$.