Chapter VII
Rank

Stability theory arose from Morley's investigation of $\aleph_1$-categorical theories, where he introduced the notion of the rank of a type in an ω-stable theory. From this beginning, Shelah and others developed a number of different rank notions to investigate more complicated (e.g. stable and superstable) theories. In this chapter we will investigate these various rank functions and their relationships to nonforking. It will become clear that our entire machine could have been constructed on the basis of rank. There are two reasons that we did not do this. In the first place rank is actually a finer tool than nonforking. We will show that for a rank function R satisfying certain axioms (in a superstable or ω-stable theory), p is a nonforking extension of q exactly if $R(p) = R(q)$. However, the rank codes additional information that is lost by the dichotomy 'forks or doesn't fork.' That is, if p is a forking extension of q we can ask how much less is $R(p)$ than $R(q)$. This additional information plays an important role in some branches of stability theory (e.g. [Cherlin, Harrington, & Lachlan 1985]). But, in the basic study of free extensions the additional information obscures the issues. More importantly, no equally simple definition of nonforking in terms of rank holds for stable theories. Rank also provides more information because it allows one to compare two types neither of which extends the other.

In Section 1 we exhibit some axioms for rank and show that any rank which satisfies these axioms induces the nonforking relation. In Section 2 we describe a number of the important rank functions. In Section 3 we prove some theorems illustrating the greater power of the rank notion.

1. Ranks and Forking

We describe here the properties that the notion of free extension satisfies when we define $p \mathcal{F} q$ if $q \subseteq p$ and $R(p) = R(q)$, where R satisfies the axioms given below. We show that these properties of the resulting freeness relation completely characterize the nonforking relation on stable theories.