Chapter IV

Finite Equivalence Relations, Definability, and Strong Types

We study in this chapter the nonforking extensions of a type $p \in S(A)$. In the first section we see that although two such types cannot be distinguished by formulas over A, they can be distinguished by formulas that are 'almost over' A. This leads to a straightforward proof of Lachlan's theorem that an $\aleph_0$-categorical superstable theory is ω-stable. In the second section we study in more detail the multiplicity of p - the number of nonforking extensions of p. We apply this analysis to reformulate the stability hierarchy in terms of definability. We introduce the important notions of the 'base' and the 'strong base' of a type. We show each type over a strongly $\kappa(T)$-saturated model is strongly based on a subset of power less than $\kappa(T)$. In Section 3 we summarise the fundamental properties of strong types. The strong type of a over A allows one to analyze the relation of a and A in terms of a stationary (i.e. multiplicity one) type. This is an essential tool for the further development of the theory. Finally we study the relation between the strong type (more generally, the multiplicity) of a pair and that of its components.

1. Finite Equivalence Relations

Unfortunately, the study of types will not provide as smooth a theory of independence as we would like. A simple example of the difficulty arises if we consider the theory of an equivalence relation with two infinite classes. Then the unique type, p, over the empty set has two extensions to a type over a model which do not fork over the empty set; namely, the type of a generic point in either equivalence class. Morley addressed this problem by saying 'the type has degree two' and the work of Lachlan [Lachlan 1975], [Lachlan 1978], successfully develops this approach. However, we find it more convenient to follow Shelah in introducing the notion of strong type. This approach describes the situation in the above example, by saying 'there are two strong types extending p', one for each equivalence class.