Chapter III

Forking

In this chapter we introduce the concept of a stable theory and expound Shelah's notion of forking. We will show that in a stable theory nonforking obeys the axioms described in Chapter II. One intuition behind this notion is that if $t(\overline{a}; B)$ is not free over A then $\overline{a}$ must satisfy more relations over B than it does over A. Each of $e, e^{\sqrt{2}}$,and $e^{1/2}$ realize the same type over Q, the field of rational numbers, namely the type p of a transcendental element. Both $p_1 = t(e^{\sqrt{2}}; Q \cup \{e\})$ and $p_2 = t(e^{1/2}; Q \cup \{e\})$ extend p, but p_1 is clearly a more generic or freer extension than p_2. We will give an account of the distinction between p_1 and p_2 which applies to any stable theory. A first approximation, '$e^{1/2}$ is in the algebraic closure of e while $e^{\sqrt{2}}$ is not,' works in a few cases. Thinking of the theory of algebraically closed fields of characteristic zero as a prototypical ω-stable theory, one can notice that the Morley rank of p_1 equals the Morley rank of p while the Morley rank of p_2 is less than that of p. This version will apply to any ω-stable theory. The extension to arbitrary stable theories requires some effort.

The most naive statement of the leitmotif of stability theory reads, 'many types implies many models; few types implies few models.' To make this notion precise we must specify what is meant by 'many types' and we must refine the phrase 'few models.' The appropriate rendering of 'few models' is 'admits a structure theory', and more specifically in this chapter, 'admits a freeness relation satisfying the axioms described in Chapter II.' We show in this chapter that a theory which has few types admits a freeness relation satisfying our axioms. We show this by introducing the notion of 'definability of types' and showing that if a theory has few types then every type is definable. From this we derive the existence of an appropriate freeness relation.

Thus in Section III.1 we show the equivalence of the two main characterizations of a stable theory.

i) There are few types, in the sense made precise in Section III.1.
ii) Every type is definable.

Thereafter, we develop the positive structure theory for stable theories solely from this definability property without further recourse to the num-