FUNCTIONAL CALCULI FOR THE LAPLACE OPERATOR IN $L^p(\mathcal{R})$

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The Laplace operator $L = -d^2/dx^2$ in $L^p(\mathcal{R})$, $1 < p < \infty$, with domain

$$\mathcal{D}(L) = \{f \in L^p(\mathcal{R});\ f' \in AC(\mathcal{R}),\ f'' \in L^p(\mathcal{R})\}$$

is a closed, densely defined operator with spectrum $\sigma(L) = [0,\infty)$; here $AC(\mathcal{R})$ is the space of functions on the real line $\mathcal{R}$ which are absolutely continuous on bounded intervals. It is known that $-L$ is the infinitesimal generator of a strongly continuous C_0-semigroup of contractions, namely the heat semigroup given by

$$(T_t f)(u) = \frac{1}{2}(\pi t)^{-1/2} \int_{-\infty}^{\infty} f(u-w)\, e^{-w^2/4t}\, dw\,, \qquad f \in L^p(\mathcal{R})\,,$$

for each $t > 0$, and that L satisfies the resolvent estimates

$$\|(L - \lambda I)^{-1}\| \leq 1/|\lambda| \sin^2\left(\frac{1}{2}\arg(\lambda)\right)\,, \qquad \lambda \in \rho(L)\,. \tag{1}$$

For $0 < \alpha < \pi$, define the open cone $S_\alpha = \{z \in \mathcal{C}\backslash\{0\};\ |\arg(z)| < \alpha\}$. A closed operator T in a Banach space X is said to be of type ω [10], where $0 \leq \omega < \pi$, if $\sigma(T) \subseteq \bar{S}_\omega$ (the bar denotes closure and, by definition, $\bar{S}_0 = [0,\infty)$) and, for $0 < \epsilon < (\pi-\omega)$, there is a positive constant c_ϵ such that

$$\|(T - \lambda I)^{-1}\| \leq c_\epsilon/|\lambda|\,, \qquad \lambda \notin \bar{S}_{\omega+\epsilon}\,.$$

It follows from (1) that if $0 < \epsilon < \pi$, then

$$\|(L - \lambda I)^{-1}\| \leq 1/|\lambda| \sin^2\left(\frac{1}{2}\epsilon\right)\,, \qquad \lambda \notin \bar{S}_\epsilon\,,$$

and hence L is of type $\omega = 0$. In particular, $-L$ then generates an analytic semigroup in the sector $\bar{S}_{\pi/2}$, [10; Theorem 3.3.1].

In the Hilbert space setting it is often the case that operators of type ω admit an $H^\infty(S_\mu)$ functional calculus for every $\omega < \mu < \pi$. For example, this is so for positive

* The support of the Centre for Mathematical Analysis (Canberra) is gratefully acknowledged. The author wishes to thank Professors M. Cowling and A. McIntosh for valuable discussions and suggestions, some of which were relevant to this work.