2. INTEGRATING GAUGES

An integration theory involves two constructions, namely that of the space of integrable function and that of the integral. These two constructions are often carried out simultaneously. However, having in mind the generalizations pursued here, it is desirable to keep them at least conceptually separated. In this chapter, spaces of integrable functions are introduced; integrals will be dealt with in the next one.

We start with a family of functions, $\mathcal{K}$, defined on a space Ω, which contains the zero-function but is not necessarily a vector space, and a non-negative real valued functional, ρ, on $\mathcal{K}$, called a gauge, such that $\rho(0) = 0$. Then we introduce the vector space $\mathcal{L} = \mathcal{L}(\rho,\mathcal{K})$ of functions, f, on Ω which can be expressed in the form

$$(*) \qquad f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for all $\omega \in \Omega$ subject to certain exceptions, where c_j are numbers and f_j functions belonging to $\mathcal{K}$, $j = 1,2,\ldots$, such that

$$(⁑) \qquad \sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty .$$

The equality $(*)$ is not required to hold for those points $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j f_j(\omega)| = \infty ,$$

even if the sum on the right in $(*)$ exists as the limit of the sequence of partial sums; the values of f at such points are arbitrary. For the seminorm, $q(f)$, of such a function f we take the infimum of the numbers (⁑). The space $\mathcal{L}$ is complete in this seminorm and the linear hull of $\mathcal{K}$ is dense in it. Of course, to avoid the obvious pathology that the seminorm of some functions $f \in \mathcal{K}$ with $\rho(f) > 0$ collapses to 0, some conditions have to be imposed on the gauge ρ. Accordingly, the gauge ρ is called integrating if $q(f) = \rho(f)$, for every function $f \in \mathcal{K}$.

If $\mathcal{K}$ is the family of characteristic functions of sets from a σ-algebra, say, and ρ is a measure on it, then this construction gives us precisely the family of functions