THE YAMABE THEOREM AND GENERAL RELATIVITY

Niall Ō Murchadha

1. INTRODUCTION

In 1960 H. Yamabe [1] claimed to have proven that every compact Riemannian manifold without boundary could be conformally deformed to one with constant scalar curvature. In 1968 Neil Trudinger [2] showed that the Yamabe proof was incorrect, and gave a partial (correct) proof. This result was improved on by T. Aubin [3] and finally completed by Richard Schoen [4] in 1984. This article consists of a rederivation and slight extension of the Schoen result and shows that the Schoen technique casts light on several problems in General Relativity.

I will consistently assume (because of my interest in the Einstein equations) that we are dealing with three-dimensional Riemannian manifolds. Much of what I do can be repeated in higher dimensions, but I will not deal with these questions.

The key thread that runs through all the analyses of the Yamabe problem is the so called Yamabe invariant of a compact manifold (M,g):

(1.1) $$Y(g) = \inf_{\theta} \frac{\int \{(\nabla\theta)^2 + \frac{1}{8}R\theta^2\}dv}{[\int \theta^6 dv]^{1/3}}$$

The Yamabe constant is a conformal invariant. Consider a conformal transformation of M by some positive function φ

(1.2) $$\bar{g}_{ab} = \varphi^4 g_{ab}$$

Given that the scalar curvature transforms as [5]

(1.3) $$\bar{R} = \varphi^{-4}R - 8\varphi^{-5}\nabla^2\varphi$$

it is easy to show (with $\bar{\theta} = \frac{\theta}{\varphi}$)