DESCRIBING SPACETIMES BY THEIR CURVATURE

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1. Introduction

In general relativity the geometry of spacetime is usually described by a Lorentz metric on a four-dimensional manifold M. In this paper we discuss the extent to which one can describe the geometry purely in terms of curvature. In Riemannian geometry this question is as old as the subject itself since it is mentioned in Riemann's inaugural lecture ([1], p.280). The word 'curvature' can be interpreted in different ways and these are not equivalent in the absence of further information concerning a metric. It could for instance mean the Riemann tensor in the form R^a_{bcd}, the same tensor in the form R_{abcd} or the sectional curvature. A more exotic possibility is to interpret it as the lasso form defined by Gross in [2]. This last interpretation is in some ways very satisfactory but has the disadvantage that one has to work with an object defined not on M itself but on its loop space. It will not be discussed further here. We will concentrate on one possibility: the Riemann tensor in the form R^a_{bcd}. (Another possibility, that of sectional curvature, was discussed in [3] and [4].)

The definition of the curvature in terms of the metric can be thought of as an inhomogeneous second order partial differential equation for the metric. The basic questions one would like to answer concerning this differential equation are the usual ones of existence, uniqueness, continuous dependence of the solution on the curvature and differentiability of the solution. By far the most difficult of these four questions is that of existence; a survey of what is known about this question for curvature quantities containing in general less information than those considered here (e.g. the Ricci tensor) can be found in the book of Kazdan[5]. The uniqueness question is equivalent to the question of whether all the information about the metric of a spacetime is contained in its curvature. This is in fact true in the generic case. Now the curvature has a physical interpretation in terms of geodesic deviation. Thus one would like to conclude from the uniqueness result that measurements of geodesic deviation provide the same information as measurements of distances and time intervals. Since, however, physical measurements are all subject to error a minimal requirement for obtaining a statement of physical significance is that a small error in the curvature produces a small error in the metric determined. In other words it is necessary to know that the determination of the metric by the curvature is continuous.

The question of differentiability is connected with the difficult issue of what degree of differentiability one should demand of physical fields. That this is not merely a matter of personal taste can be seen in the analysis of the initial value problem for Einstein's equation or, in another context, in the way that quantum theory in many cases requires the use of distributional fields since more regular ones are of measure zero. Actually the analysis which we carry out here is rather insensitive to the type of differentiability used.