SOME PROBLEMS CONCERNING REFLEXIVE OPERATOR ALGEBRAS

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1. INTRODUCTION AND PRELIMINARIES

We discuss below some problems concerning a certain class of algebras of operators on complex Banach space. Each algebra of the class arises from a lattice of subspaces of the underlying space (in a way that will soon be made precise) and most of the problems are of the form: find conditions, additional to those specified à priori, on the lattice of subspaces, which are both necessary and sufficient for the corresponding algebra of operators to have a certain, specified, algebraic and/or topological property. It is more accurate to say, then, that these problems concern lattices of subspaces of certain types. Naturally, all the problems have partial solutions some of which will be described. While the interested reader can find complete proofs elsewhere, some brief proofs are included for the purpose of illustration. Most of these problems have arisen (in some cases, have re–surfaced) in joint work with S. Argyros and M.S. Lambrou.

Throughout, X denotes a complex non–zero Banach space and H denotes a complex non–zero Hilbert space. The topological dual of X is denoted by X^*. The terms 'operator' and 'subspace' will mean bounded linear mapping and closed linear manifold, respectively. For any family $\{M_\gamma\}$ of subspaces of X, $\vee M_\gamma$ denotes the closed linear span of $\{M_\gamma\}$. For any vectors $e \in X$ and $f^* \in X^*$, $f^* \otimes e$ denotes the operator on X given by $(f^* \otimes e)\, x = f^*(x)\, e$. The lattice of subspaces of X is denoted by $\mathcal{C}(X)$ and for any $L \in \mathcal{C}(X)$, $L^\perp$ denotes the annihilator of L, that is

$$L^\perp = \{f^* \in X^* : f^*(x) = 0, \quad \text{for every } x \in L\}.$$

For any subset $\mathcal{L} \subseteq \mathcal{C}(X)$ the set of operators on X that leave every member of $\mathcal{L}$ invariant is denoted by $\operatorname{Alg} \mathcal{L}$. Thus