CONSTRUCTIONS PRESERVING WEAK AMENABILITY

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0. INTRODUCTION

When one considers module derivations from commutative Banach algebras it seems natural to restrict the attention to symmetric modules, that is, modules where left and right module multiplications agree. In particular one suspects that, if a commutative Banach algebra $\mathcal{A}$ has the property that every continuous module derivation from $\mathcal{A}$ into any symmetric Banach $\mathcal{A}$-module is zero, then $\mathcal{A}$ is particularly nice. Phrased in terms of Hochschild-Johnson cohomology groups, we require that $H^1(\mathcal{A}, X) = (0)$ for all symmetric Banach $\mathcal{A}$-modules X. Commutative Banach algebras with this property were called *weakly-amenable* (from here on abbreviated WA) in [1]. In that paper the authors related WA to amenability for certain classes of Banach algebras. They also showed that the only symmetric module which one has to consider is the dual of $\mathcal{A}$, that is, $\mathcal{A}$ is WA if and only if $H^1(\mathcal{A}, \mathcal{A}^*) = (0)$.

In a subsequent paper ([5]) the present author gave a characterization of WA in forms of the short exact sequence

$$\sum : 0 \to K \xrightarrow{i} \mathcal{A}^{\#}\hat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0,$$

where $\pi(a \otimes b) = ab$ and $K = \ker \pi$. (Here and throughout $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbf{C}$, the algebra obtained by formal adjunction of a unit.) The algebra $\mathcal{A}$ is WA if and only if $(K^2)^- = K$. If $\mathcal{A}$ has a b.a.i. then one may replace $\mathcal{A}^{\#}$ by $\mathcal{A}$. This parallels completely the corresponding characterization of amenability (Theorem III.21 of [9]). The result is especially useful for Banach algebras which behave nicely under the formation of projective tensor product.

To illustrate this let us show: