BANACH ALGEBRAS GENERATED BY ANALYTIC SEMIGROUPS HAVING COMPACTNESS PROPERTIES ON VERTICAL LINES *

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§0. INTRODUCTION

Let $(a^z)_{\mathrm{Re}\ z>0}$ be an analytic semigroup in a Banach algebra A such that $\{a^{1+iy} : y \in \mathbf{R}\}$ is relatively weakly compact in A. Then the spectrum of a^1 is countable. This result was proved in [8], in order to investigate the relationships between the structure of a locally compact group G and the existence in the group algebra $L^1(G)$ of analytic semigroups with convenient behavior on vertical lines. The proof given in [8] was based on the fact that the function $y \mapsto a^{2+iy}$, $\mathbf{R} \to A$ is a vector version of the so-called weakly almost periodic functions defined by W.E. Eberlein in 1949, and it used elementary properties of this kind of function.

In this paper we give another proof of the same result, by considering a different approach. In fact, we are able to obtain a fairly complete description of the Banach algebras which are generated by semigroups with properties as above. We show that such an algebra A is semisimple if and only if $A^{\perp} = (0)$, where $A^{\perp} = \{b \in A : bA = (0)\}$, and then, that this class of algebras is exactly the class formed by the commutative Banach algebras A which are generated by their idempotents and for which the character space is discrete and countable. Moreover, this in turn is equivalent to the existence of a one-one, bounded algebra homomorphism $\psi : \ell^1(\mu) \to A$ such that $\psi(\ell^1(\mu))^- = A$, where $\mu = \{\mu_n\}_{n=1}^{\infty}$ is a sequence of numbers greater than or equal to one. (Here $\ell^1(\mu)$ is endowed with coordinatewise operations.) So the algebras $\ell^1(\mu)$ are canonical among this kind of algebras.

The strategy carried out in the proof of the main results of the paper is as follows. If an analytic semigroup $(a^z)_{\mathrm{Re}\ z>0}$ in a Banach algebra A satisfies $[a^1A]^- = A$ and

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