31 Nitsche's Conjecture

Conjecture 31.1 (Nitsche) *Let $A \subset \mathbf{R}^3$ be an embedded complete minimal annulus such that $A \cap P_t$ is a Jordan curve for $t_1 < t < t_2$, where $-\infty \leq t_1 < t_2 \leq \infty$. Then A must be a catenoid. In particular, $t_1 = -\infty$ and $t_2 = \infty$.*

Nitsche made this conjecture in [62], it is still open. We only know that the conjecture is true under certain extra hypotheses, In this section we will give two such theorems. The first one, Theorem 31.2, is due to Nitsche [62]; the proof given here is essentially Nitsche's proof.

Theorem 31.2 *If each $A \cap P_t$ is a starshaped Jordan curve for $t_1 < t < t_2$, then A is a catenoid.*

Proof. By a translation we may assume that $t_1 < 0$, $t_2 > 0$. Let $0 < a < t_2$ and let $A \cap S(0, a)$ be a compact minimal annulus with Jordan curve boundary. By Lemma 9.1 and Proposition 9.2, its conformal structure is

$$A_{R(a)} = \{z \in \mathbf{C} \,\big|\, 1 \leq |z| \leq R(a)\},$$

where $R(a) > 1$.

Let $X(a) : A_{R(a)} \to \mathbf{R}^3$ be the conformal embedding. Then we know that the third coordinate $X(a)_3$ must be

$$X(a)_3 = \frac{a}{\log R(a)} \log |z|.$$

Let $0 < a < b < t_2$. The moduli of $A_{R(a)}$ and $A_{R(b)}$ are $R(a)$ and $R(b)$ respectively. Since $A \cap S(0, a) \subset A \cap S(0, b)$, we have $R(a) < R(b)$ and thus $A_{R(a)} \subset A_{R(b)}$. We have $X(b) : A_{R(b)} \to \mathbf{R}^3$ such that

$$X(b)_3 = \frac{b}{\log R(b)} \log |z|.$$

It must be that $X(b)_3|_{A_{R(a)}} = X(a)_3$, thus

$$\frac{b}{\log R(b)} \log R(a) = X(b)_3(R(a)e^{i\theta}) = X(a)_3(R(a)e^{i\theta}) = a,$$

which implies that

$$\frac{b}{\log R(b)} = \frac{a}{\log R(a)}. \tag{31.173}$$

Now let $0 < a_1 < a_2 < \cdots < a_n < \cdots t_2$ and $\lim_{n\to\infty} a_n = t_2$; we have $A_{R(a_1)} \subset \cdots \subset A_{R(a_n)} \subset \cdots$. Let $R = \lim_{n\to\infty} R(a_n) \leq \infty$. Then the conformal structure