30 A Generalisation of Shiffman's Second Theorem

Shiffman's second theorem says that if a minimal annulus is bounded by circles in parallel planes, then every level set is a circle.

In [25], it is proved that the same conclusion is true if we replace the boundary circles in Theorem 29.2 by parallel straight lines and assume A is properly embedded.

Furthermore, Toubiana [78] has proved that if two non-parallel straight lines lie in distinct parallel planes then they cannot bound any proper minimal annulus in the slab bounded by the planes.

In this section we will give a generalization of the results stated above, with a unified proof.

Theorem 30.1 *Suppose $A \subset S(-1,1)$ is a minimal annulus in a slab and $A(1) = A \cap P_1$, $A(-1) = A \cap P_{-1}$ are straight lines or circles.*

1. *If both $A(1)$ and $A(-1)$ are circles, then $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$. In particular, A is embedded.*
2. *If at least one of the $A(1)$ and $A(-1)$ is a straight line and A is properly embedded, then $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$.*

Remark 30.2 The first part of Theorem 30.1 is exactly Shffiman's second theorem, Theorem 29.2. We will see that the second part of theorem 30.1 implies the results in [25] and [78].

Let $A \subset S(-1,1)$ be a proper minimal annulus such that $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are straight lines or circles and $\partial A = A(1) \cup A(-1)$. In the case that there is only one straight line, we will always assume that $A(1)$ is the straight line. Then the interior of A is conformally equivalent to the interior of

$$A_R = \{z \in \mathbf{C} : 1/R \leq |z| \leq R\},$$

for some $1 < R < \infty$. In fact the interior of A is conformally equivalent to

$$\{z \in \mathbf{C} : \rho < |z| < P,\ \ 0 \leq \rho < P \leq \infty\},$$

for some ρ and P. Since A has 1-dimensional boundary ∂A which is separated by the interior of A, it follows $0 < \rho$ and $P < \infty$. Hence if $R = \sqrt{P/\rho} > 1$ then $\mathrm{Int}(A) \cong \mathrm{Int}(A_R)$.

There is a conformal harmonic immersion

$$X : A_R - C \hookrightarrow S(-1,1),$$

where C is a subset of ∂A_R and $X(\{|z| = R\} - C) = A(1)$, $X(\{|z| = 1/R\} - C) = A(-1)$. If $A(1)$ and $A(-1)$ are both circles, then $C = \emptyset$; if only $A(1)$ is a straight line, then