23 Annular Ends Lying above Catenoid Ends

The Cone Lemma (Theorem 21.1) gives a criterion for a minimal surface to have finite total curvature by looking at the picture of its image. In this section we will give another such criterion due to Fang and Meeks [18].

Consider the family of catenoids

$$C_t = \{(x, y, z) \in \mathbf{R}^3 \mid t^2x^2 + t^2y^2 = \cosh^2(tz)\},$$

for $t > 0$. We will show that a properly immersed, complete minimal annulus with one compact boundary that lies above some C_t must have finite total curvature. More precisely:

Theorem 23.1 *Let*

$$W_t = \{(x, y, z) \in \mathbf{R}^3 \mid t^2x^2 + t^2y^2 \leq \cosh^2(tz),\ z \geq 0\}.$$

Suppose $X\colon M \to \mathbf{R}^3$ is a complete, proper minimal immersion of an annulus with smooth compact boundary such that the image is contained in W_t for some $t > 0$. Then M has finite total curvature.

We will break the proof of Theorem 23.1 into several lemmas. First let us fix the notation.

Let C be a catenoid in $\mathbf{R}^3$ with the z-axis as symmetry axis. Let W be the closure of the component of $\mathbf{R}^3 - C$ that contains the z-axis. Let $\mathbf{H} = \{(x, y, z) \in \mathbf{R}^3 \mid z > 0\}$ and $\overline{\mathbf{H}}$ be its closure.

Conformally we can write $M = \{\zeta \in \mathbf{C} \,|\, 0 < r_1 \leq |\zeta| < r_2\}$. The smooth compact boundary of X corresponding to $|\zeta| = r_1$. Complete means that $X \circ \gamma$ has infinite arc length as γ diverges to $|\zeta| = r_2$. Let $A = X(M)$.

After homothetically shrinking or expanding C and A, we can assume that C is the standard catenoid, i.e., C has the conformal structure of $\mathbf{C} - \{0\}$ and is embedded in $\mathbf{R}^3$ as follows:

$$F : \mathbf{C} - \{0\} \hookrightarrow \mathbf{R}^3$$

$$F(\zeta) = \Re\left(\int_1^\zeta \omega_1, \int_1^\zeta \omega_2, \int_1^\zeta \omega_3\right) + (-1, 0, 0),$$

where

$$\omega_1 = \frac{1}{2}\frac{(1 - \zeta^2)}{\zeta^2}\, d\zeta,\quad \omega_2 = \frac{i}{2}\frac{(1 + \zeta^2)}{\zeta^2}\, d\zeta,\quad \omega_3 = \frac{d\zeta}{\zeta}.$$

The Gauss map of C is

$$N^C(\zeta) = \frac{1}{1 + |\zeta|^2}(2\Re\zeta, 2\Im\zeta, |\zeta|^2 - 1).$$

All lemmas in the following having the same assumptions as for Theorem 23.1.

The first lemma is the key point of the proof of Theorem 23.1.