21 The Cone Lemma

Let X_c be the cone in $\mathbf{R}^3$ defined by the equation

$$x_1^2 + x_2^2 = (x_3/c)^2, \quad c \neq 0.$$

The complement of X_c consists of three components, two of which are convex. We label the third region W_c and note that W_c contains $P^0 - \{0\}$, where $P^t = \{x_3 = t\}$ for $t \in \mathbf{R}$. Suppose $M \subset W_c$ is a noncompact, properly immersed minimal annulus with compact boundary.

Note that as $c \to 0$, $X_c - \{0\}$ collapses to a double covering of $P^0 - \{0\}$. Note also that any horizontal plane or vertical catenoid is eventually disjoint from any X_c, hence eventually contained in W_c, no matter how small c is (by "eventually" we mean "outside of a compact set"). Since any embedded complete minimal annular end of finite total curvature is asymptotic to a plane or a catenoid (a graph with logarithmic growth), it follows that, after suitable rotation, such an end is eventually contained in any W_c. By Jorge and Meeks' theorem, Theorem 12.1, it is easy to see that a minimally immersed end of finite total curvature with a horizontal limit tangent plane is also eventually contained in every X_c. The Cone Lemma [29] shows that this property implies that the annular end must have finite total curvature if it is proper. Hence after a rotation if necessary, a proper minimal annular end has finite total curvature if and only if it is eventually contained in every X_c.

Let $A := \{z \in \mathbf{C} \,|\, 1 \leq |z| < \infty\}$.

Theorem 21.1 (The Cone Lemma) *Let $X : A \hookrightarrow \mathbf{R}^3$ be a properly immersed minimal annulus with compact boundary. If $M := X(A)$ is eventually contained in W_c for a sufficiently small c, then X has finite total curvature.*

In order to prove the Cone Lemma we need to introduce the concept of *foliation.*

Definition 21.2 Let M be a C^∞ manifold of dimension 3. A C^k, $1 \leq k \leq \infty$, foliation of M is a set of leaves $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ in M that satisfies the following conditions:

1. $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of disjoint 2-submanifolds.

2. $\bigcup_{\alpha \in \mathcal{A}} \mathcal{L}_\alpha = M$.

3. For all points $p \in M$ there exists a neighbourhood U of M and class C^kcoordinate system (x_1, x_2, x_3) of U such that $\mathcal{L}_\alpha \cap U$ is empty or is the solution of $x_3 =$ constant in U.

Before proving Theorem 21.1, we will state a fact about the catenoid. Let C be the unit circle in P^0 centred at $(0,0)$. Let C_h be the translate of C in the plane P^h. There is an $h_2 > 0$ such that for $0 < h < h_2$ there are two catenoids bounded by C_{-h} and C_h; one is stable and the other is unstable. While the C_{-h_2} and C_{h_2} bound only one